Math. 311
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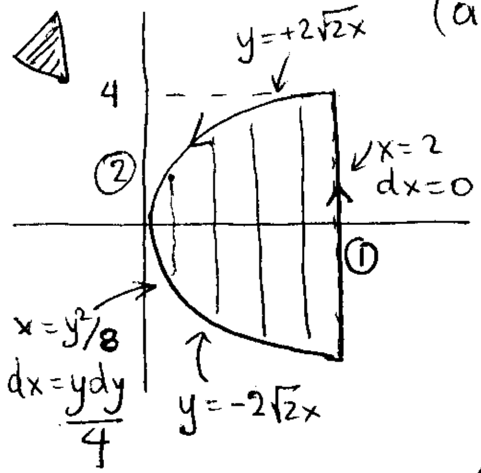
Sc' um's:
p. 2(40,42,52,54,74)
Text. p. 247 (16)

S.p.132(40) Evaluate $\oint (x^2-2xy)dx+(x^2y+3)dy$ around the boundary of the region defined by $y^2=8x$ and $x=2$ (a) directly (b) by Green's thm.



(a) $I=\oint (x^2-2xy)\,dx + (x^2y+3)\,dy =$

$$\underset{①}{\int (4y+3)dy} + \underset{②}{\int_4^{-4} \left\{\left(\frac{y^4}{64}-\frac{y^3}{4}\right)\frac{y}{4}+\left(\frac{y^5}{64}+3\right)\right\}dy}$$

① $x=2$, $dx=0$, $-4\le y\le 4$

② $x=y^2/2$, $dx=ydy$, $-4\le y\le 4$

$$= \cancel{2y^2}+\cancel{3y}\Big|_{-4}^{4} - \left(\frac{y^6}{24} - \frac{y^5}{80} + \frac{y^6}{24} + 3y\right)\Big|_{-4}^{4} = \frac{2}{80}4^5 = \frac{128}{5}$$

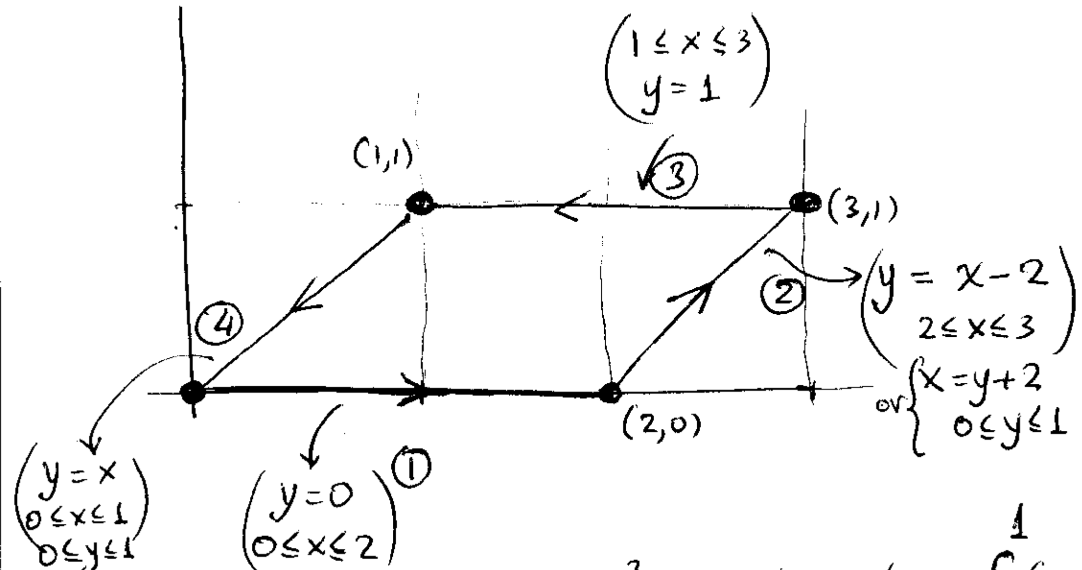
(b) $$I=\iint\left(\frac{\partial(x^2y+3)}{\partial x}-\frac{\partial(x^2-2xy)}{\partial y}\right)dxdy = \iint(2xy+2x)\,dxdy$$

$$=2\int_{x=0}^{2}dx\cdot x\int_{y=-2\sqrt{2x}}^{2\sqrt{2x}}(y+1)\,dy = 2\int_0^2 dx\cdot x\left(\frac{y^2}{2}+y\right)_{-2\sqrt{2x}}^{2\sqrt{2x}}$$

$$=8\sqrt{2}\int_0^2 \underset{x^{3/2}}{x\sqrt{x}}\,dx = 8\sqrt{2}\,\frac{2}{5}x^{5/2}\Big|_0^2 = \frac{16\sqrt{2}\,2^{5/2}}{5} = \frac{128}{5}$$

S. p.132(42) Evaluate $\oint (3x^2+2y)dx-(x+3\cos y)dy$ around the parallelogram having vertices at $(0,0)$, $(2,0)$, $(3,1)$ and $(1,1)$.

over →



① $dy = 0$, $y = 0$
② $dy = dx$
③ $dy = 0$, $y = 1$
④ $dy = dx$

$$\oint = \underset{①}{\int_{x=0}^{2} (3x^2)\,dx} + \cancel{\int_{x=2}^{3} (3x^2 + 2(x-2))} \ \underset{②}{\int_{y=0}^{1} \left(3(y+2)^2 + \underbrace{\cancel{2y}}_{0} - (y+2) + \underbrace{\cancel{3\cos y}}_{0}\right) dy}$$

$$- \underset{③}{\int_{x=1}^{3} (3x^2 + 2)\,dx} - \underset{④}{\int_{y=0}^{1} (3y^2 + \cancel{2y} - y - \cancel{3\cos y})\,dy} =$$

(note several cancellations)

$$= \int_{y=0}^{1} \left(\overbrace{3(y+2)^2 - 3y^2}^{\cancel{3y^2} + 12y + 12 - \cancel{3y^2}} - \cancel{y} - 2 + \cancel{y}\right) dy - \int_1^3 2\,dx + \int_0^2 3x^2\,dx - \int_1^3 3x^2\,dx$$

$$= 6y^2 + 10y \Big|_0^1 - 2x \Big|_1^3 + x^3 \Big|_0^2 - x^3 \Big|_1^3$$

$$= 16 - 4 + 8 - 27 + 1 = -6$$

S. p. 133 (52) Evaluate $\oiint_S F \cdot n\, dS$, $F = 2xy\,\vec{i} + yz^2\,\vec{j} + xz\,\vec{k}$

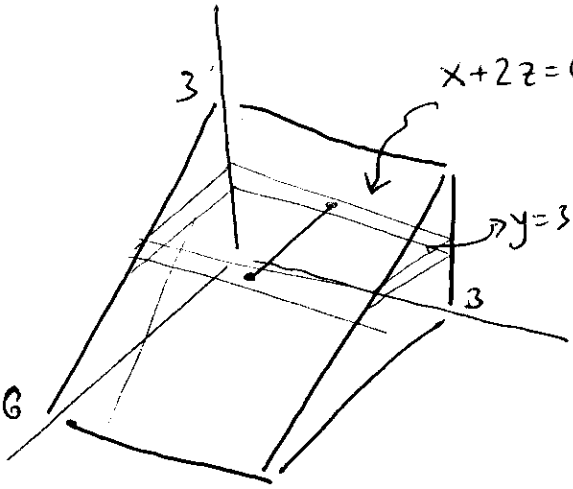
and S is:

(a) Parallelepiped:
$x = y = z = 0$
$x = 2$, $y = 1$, $z = 3$

$$\nabla \cdot F = 2y + z^2 + x$$

$$\oiint = \iiint_V (2y + z^2 + x)\,dV = x\Big|_0^2 y^2\Big|_0^1 z\Big|_0^3 + x\Big|_0^2 y\Big|_0^1 \frac{z^3}{3}\Big|_0^3 + \frac{x^2}{2}\Big|_0^2 y\Big|_0^1 z\Big|_0^3 = 6 + 18 + 6 = 30$$

(b) The surface of the region bounded by $x=0$, $y=0$ $y=3$, $z=0$ and $x+2z=6$



$$\iiint (2y+z^2+x)\,dx\,dy\,dz =$$

$$= \int_{z=0}^{3} dz \int_{x=0}^{6-2z} dx \int_{y=0}^{3} dy\,(2y+z^2+x)$$

$$= \int_{z=0}^{3} dz \int_{x=0}^{6-2z} dx \quad y^2+z^2y+xy \Big|_{y=0}^{3}$$

$$= \int_{z=0}^{3} dz \int_{x=0}^{6-2z} dx\,(9+3z^2+3x) = 3\int_{z=0}^{3} dz\,\left(3x+z^2x+\frac{x^2}{2}\right|_{x=0}^{6-2z}$$

$$= 3\int_{z=0}^{3} dz\,\left(3(6-2z)+z^2(6-2z)+\frac{1}{2}(6-2z)^2\right)$$

$$= 6\int_{z=0}^{3} dz\left[3(3-z)+z^2(3-z)+(3-z)^2\right]$$

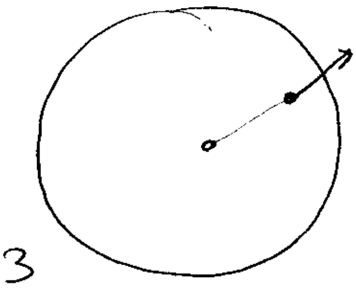
$$= 6\left(9z-\frac{3}{2}z^2+z^3-\frac{z^4}{4}-\cancel{\frac{1}{3}(3-z)^3}\right|_0^3$$

$$= 6\left(27-\frac{27}{2}+27-\frac{81}{4}+\right) = 162\left(1-\frac{1}{2}+1-\frac{3}{4}\right)+54$$

$+9$

$$= 351/2$$

S. p.133(54) Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\, dS$ where (a) S is the sphere of radius 2 with center at (0,0,0).



$\hat{n} = \dfrac{\vec{R}}{R}$, $dS = R^2 \sin\phi\, d\phi\, d\theta$

$$\iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iint_{R=2} \vec{R}\cdot\frac{\vec{R}}{R}\, R^2 \sin\phi\, d\phi\, d\theta = 2\iint dS$$

$$= 2\cdot 4\pi R^2 = 32\pi$$

$\nabla\cdot\mathbf{r} = 3$

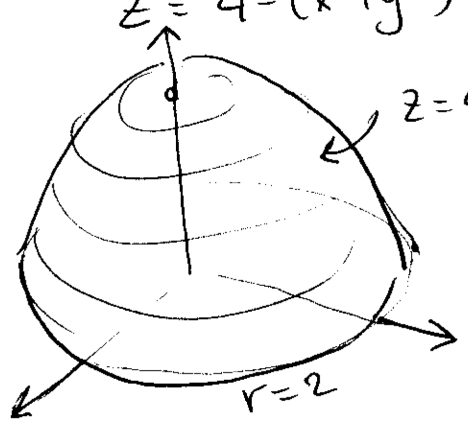
$$\left(\iiint 3\, dv = 3\cdot\frac{4}{3}\pi R^3 = 32\pi\right)$$

(b) S is the surface of the cube bounded by $x = y = z = \pm 1$.

Here it's no contest: use volume integral:

$$\oiint \mathbf{r}\cdot\mathbf{n}\, ds = \iiint \nabla\cdot\mathbf{r}\, dV = 3\iiint dV = 3\cdot 2^3 = 24$$

(c) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane



$z = 4-r^2$ Again $\oiint \mathbf{r}\cdot\mathbf{n}\, ds = 3\iiint dV$

Here, use cylindrical coords:

$$3\iiint dV = 3\int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{2} dr\, r \int_{z=0}^{4-r^2} dz$$

$$= 6\pi \int_{r=0}^{2} dr\, r(4-r^2) = 6\pi\cdot\left(2r^2 - \frac{r^4}{4}\right)\Bigg|_0^2$$

$$= 24\pi.$$