Homework #2 Solutions

Math 311

p. 51, #3: Find the area of the triangle with vertices $(1,1,2)$, $(2,3,5)$, $(1,5,5)$.

Solution: Set

$$\mathbf{A} = (2-1)\,\mathbf{i} + (3-1)\,\mathbf{j} + (5-2)\,\mathbf{k}, \quad \mathbf{B} = (1-1)\,\mathbf{i} + (5-1)\,\mathbf{j} + (5-2)\,\mathbf{k}.$$

Check that $\mathbf{A}\times\mathbf{B} = -6\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$. Since the area, A, of the triangle is $A = (1/2)|\mathbf{A}\times\mathbf{B}|$, one gets that $A = \sqrt{61}/2$. ∎

p. 51, #11: By taking the vector cross product of $\mathbf{A} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ with $\mathbf{B} = \cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j}$ and intepreting geometrically, derive a well-known geometric identity.

Solution: It is straightforward to compute that

$$\mathbf{A}\times\mathbf{B} = (\cos\theta\sin\psi - \cos\psi\sin\theta)\,\mathbf{k}.$$

Furthermore, one can check that $|\mathbf{A}| = |\mathbf{B}| = 1$. Assuming that $\psi > \theta$, the positive angle between $\mathbf{A}$ and $\mathbf{B}$ is $\phi = \psi - \theta$. By definition,

$$\begin{aligned} \mathbf{A}\times\mathbf{B} &= |\mathbf{A}|\,|\mathbf{B}|\sin\phi\,\mathbf{n} \\ &= \sin\phi\,\mathbf{n}, \end{aligned}$$

where $\mathbf{n}$ is the unit vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$ which follows the right-hand rule. Since $\phi > 0$, one has that $\mathbf{n} = \mathbf{k}$; hence, by equating the coefficients of the two expressions for $\mathbf{A}\times\mathbf{B}$ one gets that

$$\sin(\psi - \theta) = \sin\phi = \cos\theta\sin\psi - \cos\psi\sin\theta. \quad ∎$$

p. 51, #21: Express $\mathbf{B} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ as the sum of a vector parallel, plus a vector perpendicular, to $\mathbf{A} = 2\,\mathbf{i} + 4\,\mathbf{j} - 2\,\mathbf{k}$.

Solution: The vector parallel to $\mathbf{A}$, $\mathbf{B}_{\|}$, is given by

$$\mathbf{B}_{\|} = \frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\,\mathbf{A} = -\frac{1}{4}\,\mathbf{A} = -\frac{1}{2}\,\mathbf{i} - \mathbf{j} + \frac{1}{2}\,\mathbf{k}.$$

The vector perpendicular to $\mathbf{A}$, $\mathbf{B}_{\perp}$, is given by

$$\mathbf{B}_{\perp} = \mathbf{B} - \mathbf{B}_{\|} = \frac{5}{2}\,\mathbf{i} + \frac{5}{2}\,\mathbf{k}. \quad ∎$$

p. 57, #2: Find the volume of the parallelepiped whose coterminal edges are arrows representing the vectors $\mathbf{A} = 3\,\mathbf{i} + 4\,\mathbf{j}$, $\mathbf{B} = 2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$, $\mathbf{C} = 5\,\mathbf{k}$.

Solution: The volume, V, is given by the triple scalar product, i.e.,

$$V = |[\mathbf{A}, \mathbf{B}, \mathbf{C}]| = |\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}| = 5. \quad \blacksquare$$

p. 57, #6: Find the equation of the plane passing through the origin parallel to the vectors $\mathbf{A} = 3\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - \mathbf{j} + 5\,\mathbf{k}$.

Solution: A vector perpendicular to the plane, $\mathbf{n}$, is given by

$$\mathbf{n} = \mathbf{A} \times \mathbf{B} = 3\,\mathbf{i} - 17\,\mathbf{j} - 4\,\mathbf{k}.$$

The equation of the plane is then

$$3(x - 0) - 17(y - 0) - 4(z - 0) = 0,$$

i.e., $3x - 17y - 4z = 0.$ $\blacksquare$

p. 60, #11: Simplify $[\mathbf{A} \times (\mathbf{A} \times \mathbf{B})] \times \mathbf{A} \cdot \mathbf{C}$.

Solution: Upon using vector identity (1.30), it is seen that

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})\,\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\,\mathbf{B}.$$

Since $\mathbf{A} \times \mathbf{A} = \mathbf{0}$, one then gets that

$$\begin{aligned} [\mathbf{A} \times (\mathbf{A} \times \mathbf{B})] \times \mathbf{A} &= -(\mathbf{A} \cdot \mathbf{A})\,\mathbf{B} \times \mathbf{A} \\ &= (\mathbf{A} \cdot \mathbf{A})\,\mathbf{A} \times \mathbf{B}. \end{aligned}$$

Upon using the definition of the scalar triple product, one finally sees that

$$[\mathbf{A} \times (\mathbf{A} \times \mathbf{B})] \times \mathbf{A} \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{A})\,[\mathbf{A}, \mathbf{B}, \mathbf{C}]. \quad \blacksquare$$