Math. 311
Spring '99

Set 4

p.70, 2.1 (1*, 2, 3*)
p.85, 2.2 (1, 2, 3*, 5*)

p.70, 2.1.1 > Let $\vec{F}(t) = \sin t\, \vec{i} + \cos t\, \vec{j} + \vec{k}$

(a) $\frac{dF}{dt} = \cos t\, \vec{i} - \sin t\, \vec{j}$

(b) $\frac{d\vec{F}}{dt} \cdot \vec{k} = 0 \Rightarrow \frac{d\vec{F}}{dt} \parallel xy$ plane

(c) When is $\frac{dF}{dt} \parallel xz$ plane? Need $\frac{d\vec{F}}{dt} \cdot \vec{j} = -\sin t = 0$

$\Rightarrow t = k\pi,\ k = 0, \pm 1, \pm 2, \ldots$

(d) Does $\vec{F}(t)$ have constant magnitude? YES

$\|F\|^2 = \underbrace{\sin^2 t + \cos^2 t}_{=1} + 1 = 2$, constant

(e) Does $\vec{F}'(t)$ have constant magnitude? YES

$\|F'\|^2 = \cos^2 t + \sin^2 t = 1$, constant

(f) Compute $\vec{F}''(t) = \frac{d^2 F}{dt^2} = -\sin t\, \vec{i} - \cos t\, \vec{j}$.

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p.70, 2.1.3 > Find $f'(t)$ in each case:

(a) $f(t) = \underbrace{(3t\vec{i} + 5t^2\vec{j})}_{(1)} \cdot \underbrace{(t\vec{i} - \sin t\, \vec{j})}_{(2)}$

$\frac{df}{dt} = (1)' \cdot (2) + (1) \cdot (2)' = (3\vec{i} + 10t\vec{j}) \cdot (t\vec{i} - \sin t\, \vec{j}) +$

$+ (3t\vec{i} + 5t^2\vec{j}) \cdot (\vec{i} - \cos t\, \vec{j}) \Rightarrow$

$\frac{df}{dt} = 3t - 10t \sin t + 3t - 5t^2 \cos t$

(b) $f(t) = \left| 2t\vec{i} + 2t\vec{j} - \vec{k} \right| = \left(4t^2 + 4t^2 + 1\right)^{1/2}$

$$\frac{df}{dt} = \frac{1}{2}\left(8t^2+1\right)^{-1/2}(16t) = \frac{8t}{\sqrt{8t^2+1}}$$

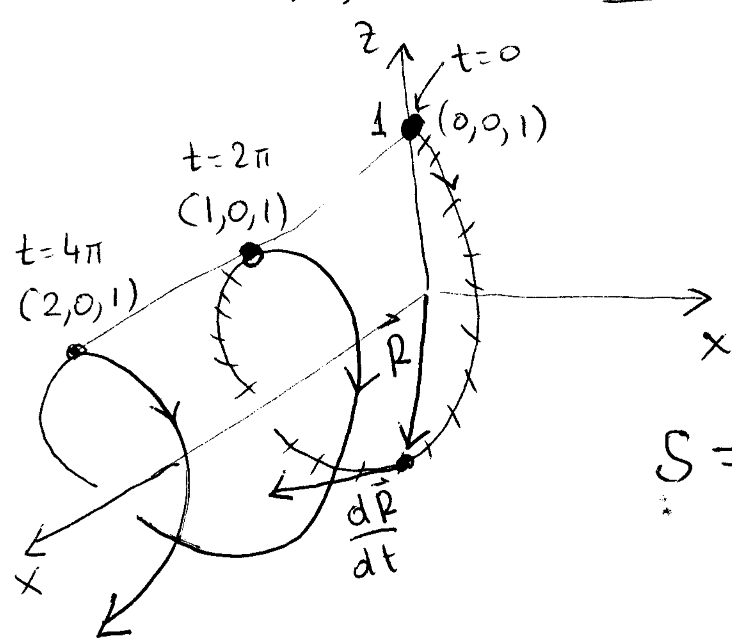
(c) $f(t) = \left[(\vec{i} + \vec{j} - 2\vec{k}) \times (3t^4\vec{i} + t\vec{j})\right] \cdot \vec{k}$

$$= \cancel{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & & \end{vmatrix}}$$

$$\frac{df}{dt} = (\vec{i} + \vec{j} - 2\vec{k}) \times (12t\vec{i} + \vec{j}) \cdot \vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 12t & 1 & 0 \end{vmatrix} \cdot \vec{k} = \begin{vmatrix} 1 & 1 \\ 12t & 1 \end{vmatrix} = 1 - 12t$$

$\boxed{\text{p. 85, 2.2.3}}$ Observe that $x = \frac{t}{2\pi}$, $y = \sin t$, $z = \cos t$ is a parametrization of the helix in example 2.14. Compute the arc length between the same two endpoints using formula (2.24). What is the unit tangent vector at $(0,0,1)$?



$$\vec{R} = \frac{t}{2\pi}\vec{i} + \sin t\,\vec{j} + \cos t\,\vec{k}$$

$$\frac{d\vec{R}}{dt} = \frac{1}{2\pi} + \cos t\,\vec{j} - \sin t\,\vec{k}$$

$$\left|\frac{d\vec{R}}{dt}\right| = \sqrt{\left(\frac{1}{2\pi}\right)^2 + \cos^2 t + \sin^2 t} = \frac{\sqrt{1+4\pi^2}}{2\pi}$$

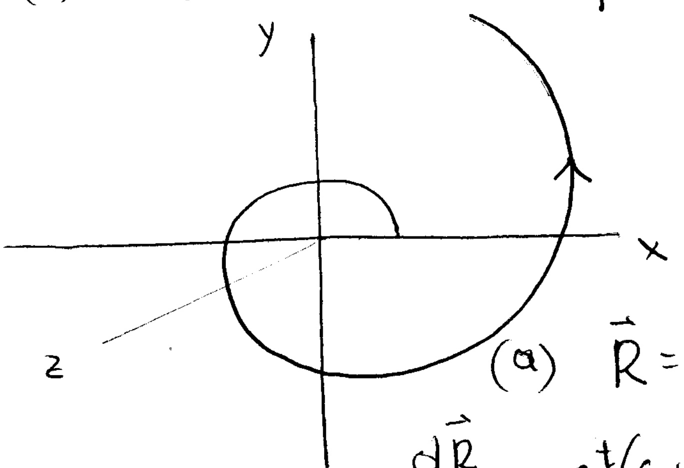
$$S = \int_{t=0}^{2\pi} \left|\frac{d\hat{R}}{dt}\right| dt = 2\pi \cdot \frac{\sqrt{1+4\pi^2}}{2\pi}$$

$$\Rightarrow \boxed{S = \sqrt{1+4\pi^2}}$$

$\boxed{\text{p.85, 2.2.5}}$ (a) Determine the arc length of the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = 0$ between $t=0$ and $t=1$.

(b) Reparametrize the curve in terms of arc length.

(c) This curve is a spiral. Sketch it to see why



(c) $\sqrt{x^2+y^2} = e^t$;

In polars : $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$

$r = e^t$, $\theta = t$

(a) $\vec{R} = e^t \cos t\, \vec{i} + e^t \sin t\, \vec{j}$

$$\frac{d\vec{R}}{dt} = e^t(\cos t - \sin t)\vec{i} + e^t(\sin t + \cos t)\vec{j}$$

$$\left|\frac{d\vec{R}}{dt}\right| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = e^t\sqrt{\cos^2 t + \sin^2 t - \cancel{2\cos t\sin t} + \sin^2 t + \cos^2 t + \cancel{2\cos t\sin t}}$$

$$= \sqrt{2}\, e^t$$

$$S = \int_{t=0}^{1} \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^t\Big|_0^1 = \sqrt{2}(e-1)$$

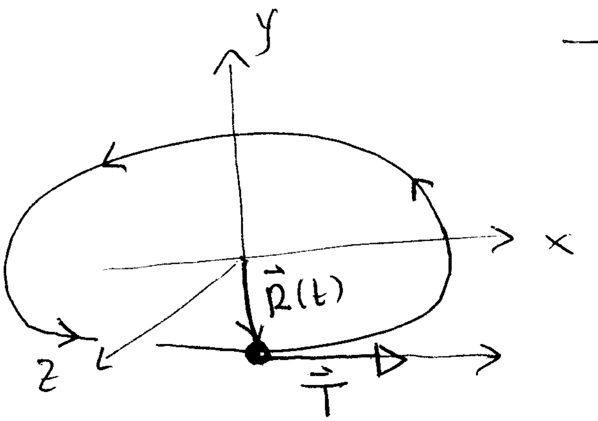
(b) Now, $S(t) = \int_0^t \sqrt{2}\, e^t\, dt = \sqrt{2}\, e^t - 1$

Solving: $e^t = \frac{S+1}{\sqrt{2}} \Rightarrow t = \ln\left(\frac{S+1}{\sqrt{2}}\right)$

Then $\left.\begin{array}{l} x = \left(\frac{S+1}{\sqrt{2}}\right)\cos\left[\ln\left(\frac{S+1}{\sqrt{2}}\right)\right] \\ y = \left(\frac{S+1}{\sqrt{2}}\right)\sin\left[\ln\left(\frac{S+1}{\sqrt{2}}\right)\right] \end{array}\right\}$ $(z=0)$

p. 85, 2.2.1 Find a unit vector tangent to the oriented closed curve

$x = a\cos t,\ y = b\sin t,\ z = 0$, at $t = \frac{3}{2}\pi$



$\vec{R} = a\cos t\ \vec{i} + b\sin t\ \vec{j}$

$\vec{V} = \frac{d\vec{R}}{dt} = -a\sin t\ \vec{i} + b\cos t\ \vec{j}$

$|\vec{V}| = (a^2\sin^2 t + b^2\cos^2 t)^{1/2}$

$$\vec{T}(t) = \frac{\vec{V}}{|V|} = \frac{-a\sin t\ \vec{i} + b\cos t\ \vec{j}}{(a^2\sin^2 t + b^2\cos^2 t)^{1/2}}$$

at $t = \frac{3\pi}{2}$ $(\cos t = 0,\ \sin t = -1)$: $\boxed{\vec{T} = \frac{a}{\sqrt{a^2}}\vec{i} = \vec{i}}$

p.85, 2.2.2 For the curve

$\{x = \sin t - t\cos t,\ y = \cos t + t\sin t,\ z = t^2\}$ find

(a) the arclength bet. $(0,1,0)$ & $(-2\pi, 1, 4\pi^2)$

(b) $\vec{T}(t)$, (c) $\vec{T}(\pi)$

$$\vec{V} = \frac{d\vec{R}}{dt} = (\cancel{\cos t} \cancel{-\cos t} + t\sin t)\vec{i} + (\cancel{-\sin t} \cancel{+\sin t} + t\cos t)\vec{j} + 2t\vec{k}$$

$v = |\vec{V}| = \sqrt{t^2\sin^2 t + t^2\cos^2 t + 4t^2} = t\sqrt{5}$:

$(0,1,0) \Rightarrow t = 0$

$(-2\pi, 1, 4\pi^2) \Rightarrow t = 2\pi$

$$\boxed{S = \int_0^{2\pi} t\sqrt{5}\,dt = \frac{\sqrt{5}}{2}t^2\Big|_0^{2\pi} = 2\sqrt{5}\pi^2}$$

$\vec{T}(t) = \frac{\vec{V}}{v} = \frac{1}{\sqrt{5}}(\sin t\ \vec{i} + \cos t\ \vec{j} + 2\vec{k})$; $\boxed{\vec{T}(\pi) = \frac{-\vec{j} + 2\vec{k}}{\sqrt{5}}}$