Math. 311
Spring '99

Set 6

p.95, 2.3 (10*, 14*, 15 (abcde)*

$\vec{V} = \frac{d\vec{R}}{dt}$; $v = \frac{ds}{dt} = \left|\frac{d\vec{R}}{dt}\right|$

$\vec{T} = \vec{V}/v$; $\frac{d\vec{T}}{ds} = k\vec{N}$

$\frac{d\vec{N}}{ds} = -\tau\vec{B}$; $\vec{B} = \vec{T}\times\vec{N}$

$a_{\parallel} = \frac{d^2s}{dt^2}$; $a_{\perp} = k\left(\frac{ds}{dt}\right)^2$

$k = |\vec{R}' \times \vec{R}''|/|\vec{R}'|^3$

p.95, 2.3.10 > A point moves along a curve so that its position is given by

$$\vec{R} = t^2\vec{i} + t^2\vec{j} + t\vec{k}$$

Find (a) speed, (b) unit tangent $\vec{T}$, (c) vector $k\vec{N}$

(a) $\vec{V} = \frac{d\vec{R}}{dt} = 2t\vec{i} + 2t\vec{j} + \vec{k}$

$$v = (4t^2 + 4t^2 + 1)^{1/2} = \sqrt{1+8t^2}$$

(b) $\vec{T} = \frac{2t\vec{i} + 2t\vec{j} + \vec{k}}{\sqrt{1+8t^2}} = \vec{V}/v$

(c) $\frac{d\vec{T}}{ds} = \frac{d}{ds}\left(\frac{\vec{V}}{v}\right) = \frac{1}{v}\frac{d}{dt}\left(\frac{\vec{V}}{v}\right)$

To economize on the algebra, we manipulate this formula a bit:

$$\frac{1}{v}\frac{d}{dt}\left(\frac{\vec{V}}{v}\right) = \frac{1}{v^2}\frac{d\vec{V}}{dt} - \frac{\vec{V}}{v^3}\frac{dv}{dt}$$

Now, $v^2 = (1+8t^2)$; $\frac{dv}{dt} = \frac{d}{dt}(1+8t^2)^{1/2} = \frac{1}{2}\cdot(1+8t^2)^{-1/2}\cdot 16t$;

finally, $\frac{d\vec{V}}{dt} = 2\vec{i} + 2\vec{j}$. So

$$k\vec{N} = \frac{d\vec{T}}{ds} = \frac{1}{(1+8t^2)}\cdot 2(\vec{i}+\vec{j}) - \frac{(2t\vec{i}+2t\vec{j}+\vec{k})}{(1+8t^2)^{3/2}}\cdot\frac{16t}{2(1+8t^2)^{1/2}}$$

$$= \frac{2(1+8t^2)(\vec{i}+\vec{j}) - 8t(2t\vec{i}+2t\vec{j}+\vec{k})}{(1+8t^2)^2}$$

$$\boxed{k\vec{N} = \frac{2(\vec{i}+\vec{j}-4t\vec{k})}{(1+8t^2)^2}}$$

6 (2/4)

p.95, 2.3.14 The position vector of a particle is given by $\vec{R}(t) = \sqrt{2}\cos 3t\,\vec{i} + \sqrt{2}\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$

Find speed, kurvature & Torsion and describe path.

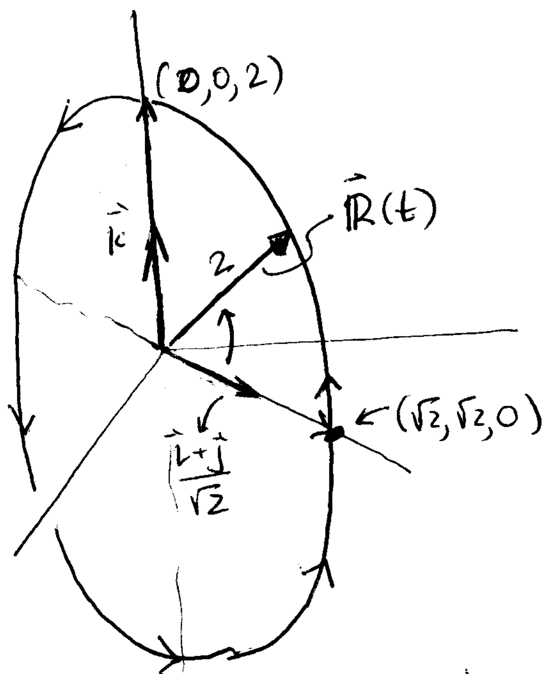
$$\vec{v} = \frac{d\vec{R}}{dt} = -3\sqrt{2}\sin 3t\,(\vec{i}+\vec{j}) + 6\cos 3t\,\vec{k}$$

$$= -6\sin 3t\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) + 6\cos 3t\,\vec{k}$$

orthogonal unit vectors

$$\vec{v} = \sqrt{(6\sin 3t)^2 + (6\cos 3t)^2} = \boxed{6 = v}$$

constant speed



path is a circle of radius 2, on plane defined by $\vec{k}$, $\frac{\vec{i}+\vec{j}}{\sqrt{2}}$

$$\vec{T} = -\sin 3t\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) + \cos 3t\,\vec{k}$$

$$\frac{d\vec{T}}{ds} = \frac{1}{v}\frac{d\vec{T}}{dt} = \frac{1}{6}\cdot\left\{-3\cos 3t\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) - 3\sin 3t\,\vec{k}\right\}$$

$$= -\frac{1}{2}\underbrace{\left\{\cos 3t\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) + \sin 3t\,\vec{k}\right\}}_{\text{unit vector}}$$

So: $\boxed{\kappa = \frac{1}{2}} \Rightarrow \rho = 2$ (circle, radius 2)

$$\vec{N} = -\left(\cos 3t\left(\frac{\vec{i}+\vec{j}}{\sqrt{2}}\right) + \sin 3t\,\vec{k}\right)$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin 3t}{\sqrt{2}} & -\frac{\sin 3t}{\sqrt{2}} & \cos 3t \\ -\frac{\cos 3t}{\sqrt{2}} & -\frac{\cos 3t}{\sqrt{2}} & -\sin 3t \end{vmatrix}$$

$$= \frac{\vec{i}}{\sqrt{2}}(\sin^2 3t + \cos^2 3t) - \frac{\vec{j}}{\sqrt{2}}(\sin^2 3t + \cos^2 3t) + \frac{\vec{k}}{2}\underbrace{(\sin 3t\cos 3t - \sin 3t\cos 3t)}_{=0}$$

$$\vec{B} = \frac{1}{\sqrt{2}}(\vec{i}-\vec{j}) \Rightarrow \frac{d\vec{B}}{ds} = \vec{0}, \quad \boxed{\tau = 0}$$

plane curve, constant curvature $\Rightarrow$ circle p.97, 2.3.15 By inspection, write down the values for each of the following:

(a) $\frac{d\vec{R}}{d\vec{s}} \circ T = \vec{T} \circ \vec{T} = 1$

(b) $\frac{d}{ds}(T \circ T) = \frac{d}{ds}(1) = 0$

(c) $\frac{d^2\vec{R}}{dt^2} \cdot \vec{T} = \frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right) \circ T = \frac{d}{dt}(v\vec{T}) \cdot \vec{T}$

$= v \underbrace{\frac{d\vec{T}}{dt} \circ \vec{T}}_{=0} + \frac{dv}{dt} T \circ T = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = a_{\parallel}$

$=0$, $\vec{T}$ unit vector, so $\frac{d\vec{T}}{dt} \perp \vec{T}$.

(d) $T \circ N = 0$

(e) $\frac{d\vec{R}}{dt} \circ \vec{T} = v\vec{T} \circ \vec{T} = v$

(f) $\frac{d\vec{N}}{ds} \circ B = (-k\vec{T} + \tau\vec{B}) \circ B = \tau$

(g) $[T, N, B] = (T \times N) \circ B = B \circ B = 1$

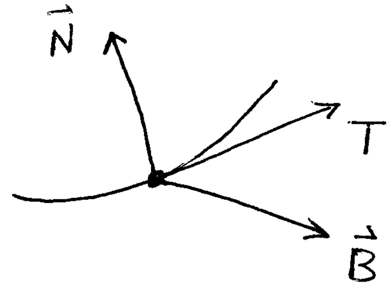
(h) $\left|\frac{d^2 R}{ds^2}\right| = \left|\frac{d\vec{T}}{ds}\right| = |k\vec{N}| = k$

(i) $\frac{d\vec{B}}{ds} = -\tau\vec{N}$

[6] (1/2)

The curvature and torsion:

$$\frac{d\vec T}{ds} = k\hat N \,;$$



Define binormal

$$\vec B = \vec T \times \vec N$$

Assume
$$\left.\begin{aligned} \frac{d\vec B}{ds} &= \alpha\vec T + \beta\vec N \\ \frac{d\vec N}{ds} &= \gamma\vec T + \delta\vec B \end{aligned}\right\}$$

$$\frac{dB}{ds} = \cancel{\alpha}\vec T + \cancel{\beta}\vec N = \frac{d\vec T}{ds}\times\vec N + \vec T\times\frac{dN}{ds}$$
$$= \underbrace{k\vec N\times\vec N}_{=0} + \vec T\times(\gamma\vec T + \delta\vec B) = -\delta\vec N$$

$$\Rightarrow \alpha = 0,\ \beta = -\delta$$

i.e. $\frac{dB}{ds} = -\delta\vec N \,;\quad \frac{d\vec N}{ds} = \gamma\vec T + \delta\vec N$

Also $\vec T = \vec N\times\vec B \Rightarrow \frac{d\vec T}{ds} = \frac{d\vec N}{ds}\times\vec B + \vec N\times\frac{d\vec B}{ds}$

$$= (\gamma\vec T + \delta\vec B)\times\vec B + \underbrace{\vec N\times(-\delta\vec N)}_{=0} = \gamma\vec N$$

$$\frac{d\vec T}{ds} = k\vec N$$

i.e. $\gamma = k$

τ : torsion

i.e.
$$\left.\begin{aligned} \frac{d\vec T}{ds} &= k\vec N \\ \frac{d\vec N}{ds} &= k\vec T + \tau\vec B \\ \frac{d\vec B}{ds} &= -\tau\vec N \end{aligned}\right\}$$

p. 97, 2.3.16 > If C is the curve given parametrically by

$$\vec{R}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$$

find
(a) The normal $\vec{N}$ and the binormal $\vec{B}$ for this curve at $t=0$.
(b) The equation of the plane passing through the point $\vec{R}(0)$ and parallel to both vectors $\vec{N}$ & $\vec{B}$ of (a).

(a) $$\frac{d\vec{R}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k} \qquad v = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$\vec{T} = \frac{1}{\sqrt{5}}\left(-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k}\right) \qquad \frac{d\vec{T}}{dt} = -\frac{1}{\sqrt{5}}\left(\cos t\,\vec{i} + \sin t\,\vec{j}\right)$$

$$\frac{d\vec{T}}{ds} = \frac{1}{v}\frac{d\vec{T}}{dt} = -\frac{1}{5}\left(\cos t\,\vec{i} + \sin t\,\vec{j}\right) : \begin{cases} K = \dfrac{1}{5} \\ \vec{N} = -(\cos t\,\vec{i} + \sin t\,\vec{j}) \end{cases}$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin t}{\sqrt{5}} & \frac{\cos t}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{2}{\sqrt{5}}\sin t\,\vec{i} - \frac{2}{\sqrt{5}}\cos t\,\vec{j} + \frac{\cos^2 t + \sin^2 t}{\sqrt{5}}\vec{k}$$

$$\boxed{\begin{aligned} \vec{T} &= \frac{-\sin t}{\sqrt{5}}\vec{i} + \frac{\cos t}{\sqrt{5}}\vec{j} + \frac{2}{\sqrt{5}}\vec{k} \\ \vec{N} &= -\cos t\,\vec{i} - \sin t\,\vec{j} \\ \vec{B} &= \frac{2\sin t}{\sqrt{5}}\vec{i} - \frac{2\cos t}{\sqrt{5}}\vec{j} + \frac{1}{\sqrt{5}}\vec{k} \end{aligned}}$$

At $t=0$:
$$\left\{ \begin{aligned} \vec{T}(0) &= \frac{1}{\sqrt{5}}(\vec{j} + 2\vec{k}) \\ \vec{N}(0) &= -\vec{i} \\ \vec{B}(0) &= \frac{1}{\sqrt{5}}(-2\vec{j} + \vec{k}) \end{aligned} \right\}$$

The plane parallel to vectors $\vec{N}, \vec{B}$ is normal to $\vec{T}$; Since $\vec{T}(0) = \frac{1}{\sqrt{5}}(\vec{j} + 2\vec{k})$; is normal.

Plane passes through $\vec{R}(0) = \vec{i} = \overrightarrow{(1,0,0)}$:

Plane: $$\vec{T}(0)\cdot\left(\vec{R} - \vec{R}(0)\right) = 0 \Rightarrow$$
$$\frac{1}{\sqrt{5}}(0,1,2)\cdot(x-1, y, z) = 0 \Rightarrow \boxed{y + 2z = 0} \rightarrow \text{plane.}$$

$\boxed{\text{p.97, 2.3.17}}$ Find the unit tangent $\vec{T}$, the principal normal $\vec{N}$, the binormal $\vec{B}$, the curvature and the torsion for

$$x = \cos^3 t, \quad y = \sin^3 t, \quad z = 2\sin^2 t, \quad (0 < t \leq \pi/2)$$

$$\mathbb{R} = \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j} + 2\sin^2 t\,\vec{k}$$

$$\vec{V} = \frac{d\mathbb{R}}{dt} = -3\cos^2 t\sin t\,\vec{i} + 3\sin^2 t\cos t\,\vec{j} + 4\sin t\cos t\,\vec{k}$$

$$V = \left(9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 16\sin^2 t\cos^2 t\right)^{1/2}$$

$$= \sin t\cos t\left(9(\cos^2 t + \sin^2 t) + 16\right)^{1/2} = \boxed{5\sin t\cos t = V}$$

$$\boxed{\vec{T} = \vec{V}/V = -\frac{3}{5}\cos t\,\vec{i} + \frac{3}{5}\sin t\,\vec{j} + \frac{4}{5}\vec{k}}$$

$$\frac{d\vec{T}}{dt} = \frac{3}{5}\sin t\,\vec{i} + \frac{3}{5}\cos t\,\vec{j}\ ; \quad \frac{1}{V}\frac{d\vec{T}}{dt} = \frac{3}{25\sin t\cos t}\underbrace{(\sin t\,\vec{i} + \cos t\,\vec{j})}_{\text{unit vector} \equiv \vec{N}}$$

$$\boxed{K = \frac{3}{25\sin t\cos t}}\ ; \quad \boxed{\vec{N} = \sin t\,\vec{i} + \cos t\,\vec{j}}$$

$$\vec{B} = \vec{T}\times\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{3}{5}\cos t & \frac{3}{5}\sin t & 4/5 \\ \sin t & \cos t & 0 \end{vmatrix} = \vec{i}\left(-\frac{4}{5}\cos t\right) + \vec{j}\left(\frac{4}{5}\sin t\right) + \vec{k}\left(-\frac{3}{5}\cos^2 t - \frac{3}{5}\sin^2 t\right)$$

$$\boxed{\vec{B} = -\frac{4}{5}\cos t\,\vec{i} + \frac{4}{5}\sin t\,\vec{j} - \frac{3}{5}\vec{k}}$$

$$\frac{d\vec{B}}{dt} = \frac{4}{5}\sin t\,\vec{i} + \frac{4}{5}\cos t\,\vec{j}\ ; \quad \frac{d\vec{B}}{ds} = \frac{1}{V}\frac{dB}{dt} = \frac{4}{25\sin t\cos t}(\sin t\,\vec{i} + \cos t\,\vec{j})$$

$$= -\tau\vec{N}$$

$$\Rightarrow \boxed{\tau = -\frac{4}{25\sin t\cos t}}$$