Math 311
Spring '99

Set 7

p. 112, 3.1 (1*, 2, 3*, 9*, 10*, 13, 14*, 20, 24*)

p. 112, 3.1.1* Compute grad f if

(a) $f = \sin x + e^{xy} + z$

$$\left.\begin{aligned} \partial_x f &= \cos x + y e^{xy} \\ \partial_y f &= x e^{xy} \\ \partial_z f &= 1 \end{aligned}\right\} \quad \vec{\nabla} f = (\cos x + y e^{xy})\vec{i} + x e^{xy}\vec{j} + \vec{k}$$

(b) $f = 1/|\vec{R}| = (x^2+y^2+z^2)^{-1/2}$

$$\left.\begin{aligned} \partial_x f &= -\tfrac{1}{2}\cdot 2x\cdot(x^2+y^2+z^2)^{-3/2} = \frac{-x}{R^3} \\ \partial_y f &= -y/R^3, \quad \partial_z f = -z/R^3 \end{aligned}\right\} \quad \boxed{\vec{\nabla} f = -\frac{\vec{R}}{R^3}}$$

very important!

(c) $f = \mathbb{R}\cdot \vec{i}\times\vec{j} = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k} = z$; $\boxed{\nabla f = \vec{k}}$

p. 112, 3.1.3 What can you say about a function whose gradient is everywhere parallel to the y-axis?

$$\vec{\nabla} f = \partial_x f\,\vec{i} + \partial_y f\,\vec{j} + \partial_z f\,\vec{k} \equiv \partial_z f\,\vec{k} : \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \end{cases}$$

i.e. $f = f(z)$ only.

p. 112, 3.1.9* Find the derivative of $f(x,y,z) = x + xyz$ at the point $(1,-2,2)$ in the direction of

(a) $2\vec{i} + 2\vec{j} - \vec{k}$; (b) $2\vec{i} + 2\vec{j} + \vec{k}$

The directional derivative of f in the direction of vector $\vec{a}$ is : $\dfrac{df}{ds} = \left(\dfrac{\vec{a}}{a}\right)\cdot\nabla f$

← unit vector in direction of $\vec{a}$.

(a)(b) $\frac{df}{ds} = \left(\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} \pm \frac{1}{3}\vec{k}\right)\cdot\left[(1+yz)\vec{i} + xz\vec{j} + xy\vec{k}\right]$

$= \frac{2}{3}(1+yz) + \frac{2}{3}xz \pm \frac{1}{3}xy$

$\vec{a} = 2\vec{i} + 2\vec{j} \pm \vec{k}\ ;\ |\vec{a}| = (4+4+1)^{1/2} = 3$

$\vec{\nabla} f = (1+yz)\vec{i} + xz\vec{j} + xy\vec{k}$

At $(1,-2,2)$: $\left.\frac{df}{ds}\right|_{(1,-2,2)} = \frac{2}{3}\left(1+2\cdot(-2)\right) + \frac{2}{3}(1\cdot 2) \pm \frac{1}{3}(1)(-2)$

$= -2 + \frac{4}{3} \pm \left(-\frac{2}{3}\right) = -\frac{2}{3} \pm \left(-\frac{2}{3}\right) \begin{cases} \xrightarrow{(+)} -4/3 \\ \xrightarrow{(-)} 0 \end{cases}$

$\boxed{\text{p. 112, 3.1,10}}$ Find the directional derivative $\frac{df}{ds}$ at $(1,3,-2)$ in the direction of $-\vec{i}+2\vec{j}+2\vec{k}$ if

(a) $f(x,y,z) = yz + xy + xz$, (b) $f(x,y,z) = x^2 + 2y^2 + 3z^2$

(c) $f(x,y,z) = xy + x^3y^3$, (d) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

(a) $\nabla f|_{(1,3,-2)} = (y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k}\Big|_{(1,3,-2)} = \vec{i} - \vec{j} + 4\vec{k}$

(b) $\nabla f|_{(1,3,-2)} = 2x\vec{i} + 4y\vec{j} + 6z\vec{k}\Big|_{(1,3,-2)} = 2\vec{i} + 12\vec{j} - 12\vec{k}$

(c) $\nabla f|_{(1,3,-2)} = (y+3x^2y^3)\vec{i} + (x+3x^3y^2)\vec{j}\Big|_{(1,3,-2)} = 84\vec{i} + 28\vec{j}$

(d) $\nabla f|_{(1,3,-2)} = \frac{x}{R}\vec{i} + \frac{y}{R}\vec{j} + \frac{z}{R}\vec{k}\Big|_{(1,-2,3)} = \frac{1}{\sqrt{14}}(\vec{i} - 2\vec{j} + 3\vec{k})$

Unit vector: $\frac{\vec{a}}{a} = \frac{-\vec{i}+2\vec{j}+2\vec{k}}{\sqrt{1+4+4}} = -\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$ $\Rightarrow$

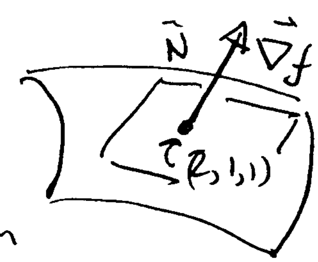
(a) $\frac{\vec{a}}{a}\cdot\nabla f\Big| = \left(-\frac{1}{3}\vec{i}+\frac{2}{3}\vec{j}+\frac{2}{3}\vec{k}\right)\cdot\left(\vec{i}-\vec{j}+4\vec{k}\right) = 5/3$

(b) $\frac{\vec{a}}{a}\cdot\nabla f\Big| = \left(\quad '' \quad\right)\cdot\left(2\vec{i}+12\vec{j}-12\vec{k}\right) = -\frac{2}{3}$

(c) $\frac{\vec{a}}{a}\cdot\nabla f = \left(\quad '' \quad\right)\cdot\left(84\vec{i}+28\vec{j}\right) = -28/3$

(d) $\frac{\vec{a}}{a}\cdot\nabla f = \left(\quad '' \quad\right)\cdot\frac{1}{\sqrt{14}}\left(\vec{i}-2\vec{j}+2\vec{k}\right) = -\frac{1}{3\sqrt{14}}$

P. 113, 3.1.14*) Find a vector normal to the surface $x^2+yz=5$ at $(2,1,1)$.



Consider $\phi(x,y,z)=x^2+yz$; then

$\vec{\nabla}\phi = 2x\vec{i}+z\vec{j}+y\vec{k}$ gives direction of normal to surface at any point. At $(2,1,1)$

$\vec{N} = \vec{\nabla}\phi\Big|_{(2,1,1)} = 4\vec{i}+\vec{j}+\vec{k}$

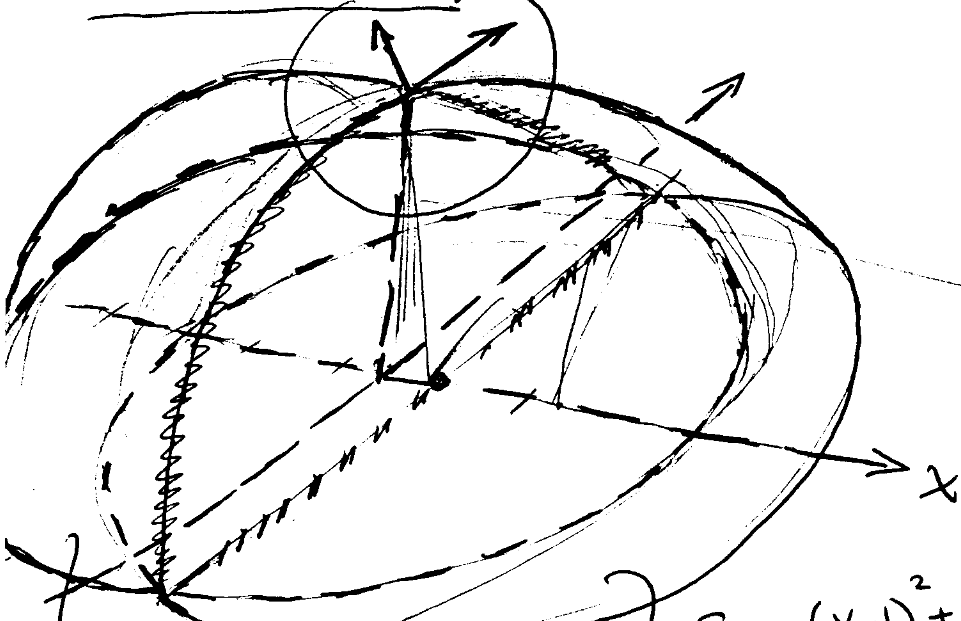
(Althoug not asked for, equ. of tangent plane:)

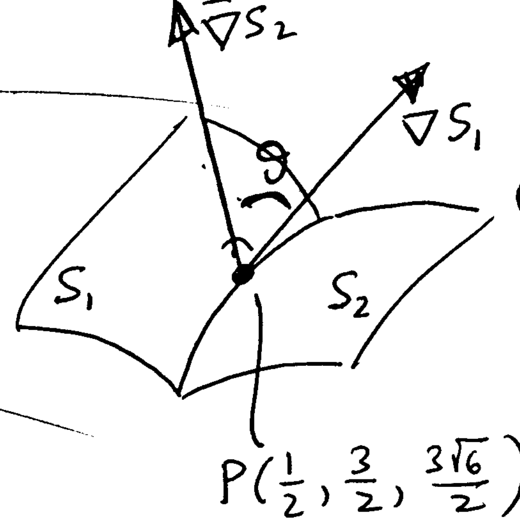
$(\vec{R}-\vec{R}_0)\cdot\vec{N}=0$; $\left\{\begin{array}{l}\vec{R}_0 = 2\vec{i}+\vec{j}+\vec{k}\\ \vec{N}=4\vec{i}+\vec{j}+\vec{k}\end{array}\right\}$

$4(x-2)+(y-1)+(z-1)=0$

or $\boxed{4x+y+z=10}$

p.112, 3.1.24*





$S_2 := (x-1)^2 + y^2 + z^2 = 16$

$S_1 := x^2 + y^2 + z^2 = 16$

$$\nabla S_1 = 2(x\vec{i} + y\vec{j} + z\vec{k})\Big|_{(\frac{1}{2}, \frac{3}{2}, \frac{3\sqrt{6}}{2})} = \vec{i} + 3\vec{j} + 3\sqrt{6}\vec{k}$$

$$\nabla S_2 = 2((x-1)\vec{i} + y\vec{j} + z\vec{k})\Big|_{(\frac{1}{2}, \frac{3}{2}, \frac{3\sqrt{6}}{2})} = -\vec{i} + 3\vec{j} + 3\sqrt{6}\vec{k}$$

Then $|\nabla S_1| = (1 + 9 + 6\cdot 9)^{1/2} = \sqrt{64} = 8$

$|\nabla S_2| = (1 + 9 + 6\cdot 9)^{1/2} = \ldots = 8$

$$\cos\theta = \frac{\nabla S_1 \cdot \nabla S_2}{|\nabla S_1||\nabla S_2|} = \frac{-1 + 9 + 54}{64} = \frac{62}{64} = \frac{31}{32}$$

$$\theta = \cos^{-1}\left(\frac{31}{32}\right) = 14.36^\circ$$