Ma311.
Spring 99
Set 8

p.117, 3.2.2*
p.124, (4*, 6*)

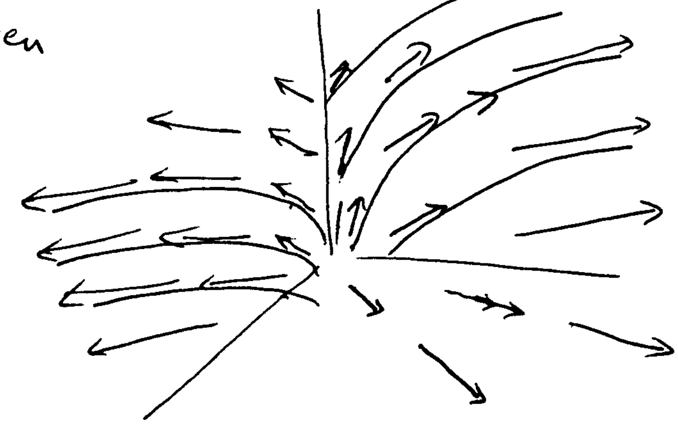
p.117, 3.2.2 > Let $F = x^2\vec{i} + y^2\vec{j} + \vec{k}$

(a) Find the general equation of a flow line.

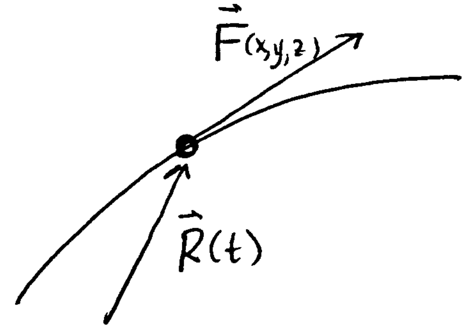
(b) Find the flow line through the point (1,1,2).

(a) A flow line is given by $\vec{R} = \vec{R}(t)$, where

$$\frac{d\vec{R}}{dt} = \vec{F} \Rightarrow$$



$$\frac{dx}{dt} = x^2, \quad \frac{dy}{dt} = y^2, \quad \frac{dz}{dt} = 1$$



i.e. $\cancel{x \frac{1}{x^2} \frac{dx}{dt}} =$

$$\frac{dx}{x^2} = dt \Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} = t + C$$

$$t=0: \; -\frac{1}{x_0} = C \Rightarrow$$

or

$$\boxed{x(t) = \frac{-1}{t - \frac{1}{x_0}} = \frac{x_0}{1 - x_0 t}}$$

$C \neq$.

Similarly: $\int \frac{dy}{y^2} = \int dt \Rightarrow y(t) = \frac{y_0}{1 - y_0 t}$

$$\int dz = \int dt \Rightarrow z = t + C \Rightarrow z(t) = t + z_0$$

(b) i.e The general equation of the flow line through (x_0, y_0, z_0): $(x(t), y(t), z(t)) = \left(\frac{x_0}{1-x_0 t}, \frac{y_0}{1-y_0 t}, t+z_0\right)$

If $x_0 = 1, y_0 = 1, z_0 = 2$:

$$\vec{R}(t) = \left(\frac{1}{1-t}\right)\vec{i} + \left(\frac{1}{1-t}\right)\vec{j} + (t+2)\vec{k}, \quad (t < 1).$$

p.124, 3.3.4 Find the divergence of the field:

$$\vec{F} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}} \quad ; \text{ write } R=(x^2+y^2+z^2)^{1/2}$$

$$\vec{\nabla}\cdot\vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$$

$$\partial_x F_1 = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) = \frac{1}{(\;\;\;)^{3/2}} + x\cdot\left(-\tfrac{3}{2}\right)\cdot(x^2+y^2+z^2)^{-5/2}\cdot(2x)$$

$$= \frac{1}{R^3} - \frac{3x^2}{R^5} = \frac{R^2-3x^2}{R^5} \quad (R\neq 0)$$

Similarly, $\partial_y F_2 = \dfrac{R^2-3y^2}{R^5}$, $\partial_z F_3 = \dfrac{R^2-3z^2}{R^5}$ $(R\neq 0)$

So: $\vec{\nabla}\cdot\vec{F} = \dfrac{R^2-3x^2}{R^5}+\dfrac{R^2-3y^2}{R^5}+\dfrac{R^2-3z^2}{R^5} = \dfrac{3R^2-3(x^2+y^2+z^2)}{R^5} = 0$

$$\left(\vec{\nabla}\cdot\vec{F}=0,\ R\neq 0\right)$$

p.124, 3.3.6 Construct an example of a scalar field ϕ and a vector field $\vec{F}$, neither of which is constant, for which $\mathrm{div}(\phi\vec{F})$ is identically equal to $\phi\,\mathrm{div}\,\vec{F}$.

Since $\nabla\cdot(\phi\vec{F}) = (\vec{\nabla}\phi)\cdot\vec{F} + \phi\,\vec{\nabla}\cdot\vec{F}$, we want to find a ϕ such that $\vec{\nabla}\phi \perp \vec{F}$. The simplest such example is $\phi=\phi(x,y)$, $\vec{F} = f(x,y,z)\,\vec{k}$

Then $\nabla\phi = \partial_x\phi\,\vec{i} + \partial_y\phi\,\vec{j}$, so that

$\nabla\phi\cdot F = 0$; say $\phi = xy$, $\vec{F} = (x^2+y^2+z^2)\vec{k}$

Then $\nabla\phi\cdot\vec{F} = (\vec{i}y+\vec{j}x)\cdot(x^2+y^2+z^2)\vec{k} = 0$