Math. 311 Spring 99

$\P$ 9

p. 132, 3.4 (1,2,3, 4*, 9*, 10*

p. 132, 3.4.4 > Given $\vec{F} = (x + xz^2)\vec{i} + xy\vec{j} + yz\vec{k}$ evaluate

(a) $\vec{\nabla} \cdot \vec{F} = \partial_x(x + xz^2) + \partial_y(xy) + \partial_z(yz)$

$$\boxed{\nabla \cdot F = (1 + z^2) + x + y}$$

(b) $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x(1+z^2) & xy & yz \end{vmatrix} = \begin{matrix} \vec{i}\,[\partial_y(yz) - \partial_z(xy)] \\ -\vec{j}\,[\partial_x(yz) - \partial_z(x + xz^2)] \\ +\vec{k}\,[\partial_x(xy) - \partial_y(x + xz^2)] \end{matrix}$

$$\Rightarrow \boxed{\nabla \times \vec{F} = \vec{i}\, z + \vec{j}\,(2xz) + \vec{k}\, y}$$

p. 132, 3.4.9 > Find a vector field whose curl is $\begin{Bmatrix} x\vec{i} \\ y\vec{i} \end{Bmatrix}$?

$$\nabla \times \vec{F} = (\partial_y F_3 - \partial_z F_2)\vec{i} - (\partial_x F_3 - \partial_z F_1)\vec{j} + (\partial_x F_2 - \partial_y F_1)\vec{k}$$

(a) Need:

$$\partial_y F_3 - \partial_z F_2 = x$$
$$\partial_x F_3 - \partial_z F_1 = 0 \Rightarrow F_3(x,y,z) = \int^x (\partial_z F_1)\,dx + \phi_3(y,z)$$
$$\partial_x F_2 - \partial_y F_1 = 0 \Rightarrow F_2(x,y,z) = \int^x (\partial_z F_2)\,dx + \phi_2(y,z)$$

$$\Rightarrow \int^x \underbrace{\{\partial_{yz} F_1 - \partial_{zy} F_1\}}_{=0}\,dx + \left(\partial_y \phi_3(y,z) - \partial_z \phi_2(y,z)\right) = x$$

Impossible!

(b) Need:

$$\left.\begin{aligned} \partial_y F_3 - \partial_z F_2 &= y \\ \partial_x F_3 - \partial_z F_1 &= 0 \\ \partial_x F_2 - \partial_y F_1 &= 0 \end{aligned}\right\}$$

Now:

$$\partial_y \phi_3 - \partial_z \phi_2 = y$$

Let: $F_1 = 0$, $F_2 = 0$, $F_3 = \phi_3(y,z)$; $\frac{\partial \phi_3}{\partial y} = y \Rightarrow \phi_3 = \frac{1}{2}y^2$

i.e. $\boxed{\vec{F} = \frac{1}{2}y^2\vec{k}}$

P. 132, 3.4.10 Given $\vec{F} = y^2\vec{i} + z^2\vec{j} + x\vec{k}$, find

(a) The curl of $\vec{F}$

(b) the component of curl $\vec{F}$ along the tangent to the curve

$x = \cos \pi t$, $y = \sin \pi t$, $z = t^2$ $(t=1)$

(a) $\vec{\nabla} \times \vec{F} = \vec{i}\left(\partial_y F_3 - \partial_z F_2\right) - \vec{j}\left(\partial_x F_3 - \partial_z F_1\right) + \vec{k}\left(\partial_x F_2 - \partial_y F_1\right)$

$= \vec{i}(-2z) - \vec{j}(1-0) + \vec{k}(-2y) = -(2z\vec{i} + \vec{j} + 2y\vec{k})$

(b) $\vec{R} = \cos\pi t\,\vec{i} + \sin\pi t\,\vec{j} + t^2\vec{k}$; $t=1$, $\vec{R}(1) = -\vec{i} + \vec{k}$

$\frac{d\vec{R}}{dt} = -\pi \sin\pi t\,\vec{i} + \pi\cos\pi t\,\vec{j} + 2t\vec{k}$

$\vec{V}(1) = -\pi\vec{j} + 2\vec{k}$; $|\vec{V}(1)| = \sqrt{\pi^2+4}$

$\vec{T}(1) = -\frac{\pi}{\sqrt{\pi^2+4}}\vec{j} + \frac{2}{\sqrt{\pi^2+4}}\vec{k}$

$\cancel{\vec{T} \cdot \nabla \times \vec{F}}$ At $t=1$, $\vec{R} = (-\vec{i} + \vec{k})$ $(x=-1, y=0, z=1)$

$\nabla \times F\Big|_{(-1,0,1)} = -2\vec{i} + \vec{j}$

$\vec{T} \cdot (\vec{\nabla} \times \vec{F}) = \frac{1}{\sqrt{\pi^2+4}}(-\pi\vec{j} + 2\vec{k})(-2\vec{i} + \vec{j}) = \frac{2\pi + 2}{\sqrt{\pi^2+4}}$