$$(2)\qquad \iiint_V \frac{\partial A_1}{\partial x}\,dV = \iint_S A_1\,\mathbf{i}\cdot\mathbf{n}\,dS$$

$$(3)\qquad \iiint_V \frac{\partial A_2}{\partial y}\,dV = \iint_S A_2\,\mathbf{j}\cdot\mathbf{n}\,dS$$

Adding (1), (2) and (3),

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) dV = \iint_S (A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k})\cdot\mathbf{n}\,dS$$

or

$$\iiint_V \nabla\cdot\mathbf{A}\,dV = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS$$

The theorem can be extended to surfaces which are such that lines parallel to the coordinate axes meet them in more than two points. To establish this extension, subdivide the region bounded by S into subregions whose surfaces do satisfy this condition. The procedure is analogous to that used in Green's theorem for the plane.

17. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ and S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$.

By the divergence theorem, the required integral is equal to

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - y)\,dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 2z^2 - yz\,\Big|_{z=0}^{1}\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx = \frac{3}{2}$$

The surface integral may also be evaluated directly as in Problem 23, Chapter 5.

18. Verify the divergence theorem for $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z=0$ and $z=3$.

$$\text{Volume integral} = \iiint_V \nabla\cdot\mathbf{A}\,dV = \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\,dV = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3} (4-4y+2z)\,dz\,dy\,dx = 84\pi$$

The surface S of the cylinder consists of a base S_1 $(z=0)$, the top S_2 $(z=3)$ and the convex portion S_3 $(x^2+y^2=4)$. Then

$$\text{Surface integral} = \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 + \iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 + \iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3$$

On $S_1\ (z=0)$, $\mathbf{n}=-\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j}$ and $\mathbf{A}\cdot\mathbf{n} = 0$, so that $\displaystyle\iint_{S_1} \mathbf{A}\cdot\mathbf{n}\,dS_1 = 0$.

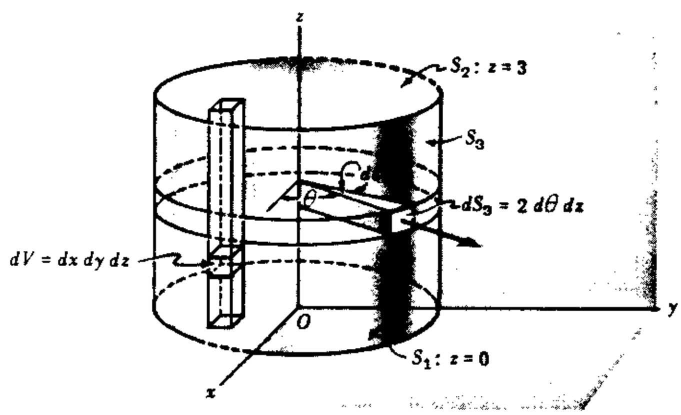
On $S_2\ (z=3)$, $\mathbf{n}=\mathbf{k}$, $\mathbf{A} = 4x\,\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$ and $\mathbf{A}\cdot\mathbf{n} = 9$, so that

$$\iint_{S_2} \mathbf{A}\cdot\mathbf{n}\,dS_2 = 9\iint_{S_2} dS_2 = 36\pi, \qquad \text{since area of } S_2 = 4\pi$$

On $S_3\ (x^2+y^2=4)$. A perpendicular to $x^2+y^2=4$ has the direction $\nabla(x^2+y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Then a unit normal is $\mathbf{n} = \dfrac{2x\,\mathbf{i}+2y\,\mathbf{j}}{\sqrt{4x^2+4y^2}} = \dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{2}$ since $x^2+y^2=4$.

$$\mathbf{A}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k})\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}}{2}\right) = 2x^2 - y^3$$



From the figure above, $x = 2\cos\theta$, $y = 2\sin\theta$, $dS_3 = 2\,d\theta\,dz$ and so

$$\iint_{S_3} \mathbf{A}\cdot\mathbf{n}\,dS_3 = \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi} (48\cos^2\theta - 48\sin^3\theta)\,d\theta = \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi$$

Then the surface integral $= 0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Note that evaluation of the surface integral over S_3 could also have been done by projection of S_3 on the xz or yz coordinate planes.

19. If div $\mathbf{A}$ denotes the divergence of a vector field $\mathbf{A}$ at a point P, show that

$$\operatorname{div}\mathbf{A} = \lim_{\Delta V\to 0} \frac{\displaystyle\iint_{\Delta S} \mathbf{A}\cdot\mathbf{n}\,dS}{\Delta V}$$

where ΔV is the volume enclosed by the surface ΔS and the limit is obtained by shrinking ΔV to the point P.

Thus by addition,

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \oint_C \mathbf{A}\cdot d\mathbf{r}$$

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, \ldots S_k$ with boundaries $C_1, C_2, \ldots C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, \ldots C_k$, the line integral over C is obtained.

32. Verify Stokes' theorem for $\mathbf{A} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

Also,

$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

Then

$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes' theorem is verified.

33. Prove that a necessary and sufficient condition that $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ for every closed curve C is that $\nabla\times\mathbf{A} = \mathbf{0}$ identically.

Sufficiency. Suppose $\nabla\times\mathbf{A} = \mathbf{0}$. Then by Stokes' theorem

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = 0$$

Necessity. Suppose $\oint_C \mathbf{A}\cdot d\mathbf{r} = 0$ around every closed path C, and assume $\nabla\times\mathbf{A} \neq \mathbf{0}$ at some point P. Then assuming $\nabla\times\mathbf{A}$ is continuous there will be a region with P as an interior point, where $\nabla\times\mathbf{A} \neq \mathbf{0}$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point has the same direction as $\nabla\times\mathbf{A}$, i.e. $\nabla\times\mathbf{A} = \alpha\mathbf{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem

SUPPLEMENTARY PROBLEMS

37. Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region defined by: (a) $y = \sqrt{x}$, $y = x^2$; (b) $x = 0$, $y = 0$, $x + y = 1$.
Ans. (a) common value = 3/2 (b) common value = 5/3

38. Evaluate $\oint_C (3x + 4y)\,dx + (2x - 3y)\,dy$ where C, a circle of radius two with center at the origin of the xy plane, is traversed in the positive sense. *Ans.* -8π

39. Work the previous problem for the line integral $\oint_C (x^2 + y^2)\,dx + 3xy^2\,dy$. *Ans.* 12π

40. Evaluate $\oint (x^2 - 2xy)\,dx + (x^2 y + 3)\,dy$ around the boundary of the region defined by $y^2 = 8x$ and $x = 2$ (a) directly, (b) by using Green's theorem. *Ans.* 128/5

41. Evaluate $\int_{(0,0)}^{(\pi,2)} (6xy - y^2)\,dx + (3x^2 - 2xy)\,dy$ along the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.
Ans. $6\pi^2 - 4\pi$

42. Evaluate $\oint (3x^2 + 2y)\,dx - (x + 3\cos y)\,dy$ around the parallelogram having vertices at (0,0), (2,0), (3,1) and (1,1). *Ans.* -6

43. Find the area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$, and the x axis.
Ans. $3\pi a^2$

44. Find the area bounded by the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$.
Hint: Parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$. *Ans.* $3\pi a^2/8$

45. Show that in polar coordinates (ρ, ϕ) the expression $x\,dy - y\,dx = \rho^2\,d\phi$. Interpret $\frac{1}{2}\int x\,dy - y\,dx$.

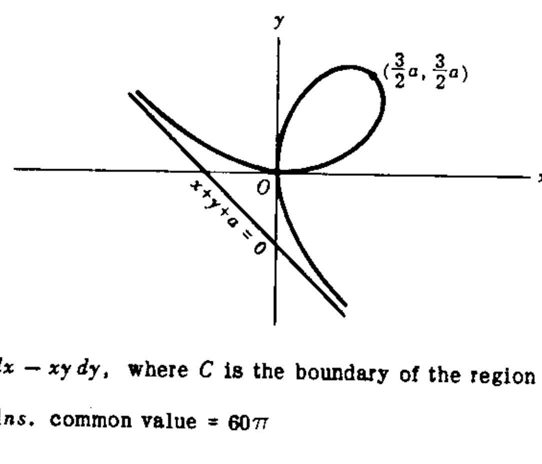
46. Find the area of a loop of the four-leafed rose $\rho = 3\sin 2\phi$. *Ans.* $9\pi/8$

47. Find the area of both loops of the lemniscate $\rho^2 = a^2\cos 2\phi$. *Ans.* a^2

48. Find the area of the loop of the folium of Descartes $x^3 + y^3 = 3axy$, $a > 0$ (see adjoining figure).
Hint: Let $y = tx$ and obtain the parametric equations of the curve. Then use the fact that

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}\oint x\,dy - y\,dx \\ &= \tfrac{1}{2}\oint x^2\,d\left(\tfrac{y}{x}\right) \\ &= \tfrac{1}{2}\oint x^2\,dt \end{aligned}$$

Ans. $3a^2/2$



49. Verify Green's theorem in the plane for $\oint_C (2x - y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. *Ans.* common value = 60π

50. Evaluate $\int_{(1,0)}^{(-1,0)} \dfrac{-y\,dx + x\,dy}{x^2 + y^2}$ along the following paths:

(a) straight line segments from (1,0) to (1,1), then to (−1,1), then to (−1,0).
(b) straight line segments from (1,0) to (1,−1), then to (−1,−1), then to (−1,0).
Show that although $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the line integral *is dependent* on the path joining (1,0) to (−1,0) and explain.
Ans. (a) π (b) $-\pi$

51. By changing variables from (x,y) to (u,v) according to the transformation $x = x(u,v)$, $y = y(u,v)$, show that the area A of a region R bounded by a simple closed curve C is given by

$$A = \iint_R \left| J\left(\frac{x,y}{u,v}\right) \right| du\,dv \qquad \text{where} \qquad J\left(\frac{x,y}{u,v}\right) \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of x and y with respect to u and v. What restrictions should you make? Illustrate the result where u and v are polar coordinates.
Hint: Use the result $A = \frac{1}{2}\int x\,dy - y\,dx$, transform to u,v coordinates and then use Green's theorem.

52. Evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F} = 2xy\,\mathbf{i} + yz^2\,\mathbf{j} + xz\,\mathbf{k}$ and S is:
(a) the surface of the parallelepiped bounded by $x=0$, $y=0$, $z=0$, $x=2$, $y=1$ and $z=3$,
(b) the surface of the region bounded by $x=0$, $y=0$, $y=3$, $z=0$ and $x+2z=6$.
Ans. (a) 30 (b) 351/2

53. Verify the divergence theorem for $\mathbf{A} = 2x^2y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ taken over the region in the first octant bounded by $y^2+z^2 = 9$ and $x = 2$. *Ans.* 180

54. Evaluate $\iint_S \mathbf{r}\cdot\mathbf{n}\,dS$ where (a) S is the sphere of radius 2 with center at (0,0,0), (b) S is the surface of the cube bounded by $x=-1$, $y=-1$, $z=-1$, $x=1$, $y=1$, $z=1$, (c) S is the surface bounded by the paraboloid $z = 4-(x^2+y^2)$ and the xy plane. *Ans.* (a) 32π (b) 24 (c) 24π

55. If S is any closed surface enclosing a volume V and $\mathbf{A} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, prove that $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = (a+b+c)V$.

56. If $\mathbf{H} = \operatorname{curl}\mathbf{A}$, prove that $\iint_S \mathbf{H}\cdot\mathbf{n}\,dS = 0$ for any closed surface S.

57. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface of area S, show that $\iiint_V \operatorname{div}\mathbf{n}\,dV = S$.

58. Prove $\displaystyle\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS$.

59. Prove $\displaystyle\iint_S r^5\,\mathbf{n}\,dS = \iiint_V 5r^3\,\mathbf{r}\,dV$.

60. Prove $\displaystyle\iint_S \mathbf{n}\,dS = \mathbf{0}$ for any closed surface S.

61. Show that Green's second identity can be written $\displaystyle\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S \left(\phi\frac{d\psi}{dn} - \psi\frac{d\phi}{dn}\right)dS$

62. Prove $\displaystyle\iint_S \mathbf{r}\times d\mathbf{S} = \mathbf{0}$ for any closed surface S.

63. Verify Stokes' theorem for $\mathbf{A} = (y-z+2)\mathbf{i} + (yz+4)\mathbf{j} - xz\,\mathbf{k}$, where S is the surface of the cube $x=0$, $y=0$, $z=0$, $x=2$, $y=2$, $z=2$ above the xy plane. *Ans.* common value $= -4$

64. Verify Stokes' theorem for $\mathbf{F} = xz\,\mathbf{i} - y\,\mathbf{j} + x^2y\,\mathbf{k}$, where S is the surface of the region bounded by $x=0$, $y=0$, $z=0$, $2x+y+2z=8$ which is not included in the xz plane. *Ans.* common value $= 32/3$

65. Evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, where $\mathbf{A} = (x^2+y-4)\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\mathbf{k}$ and S is the surface of (*a*) the hemisphere $x^2+y^2+z^2=16$ above the xy plane, (*b*) the paraboloid $z = 4-(x^2+y^2)$ above the xy plane. *Ans.* (*a*) -16π, (*b*) -4π

66. If $\mathbf{A} = 2yz\,\mathbf{i} - (x+3y-2)\mathbf{j} + (x^2+z)\mathbf{k}$, evaluate $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$ over the surface of intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant. *Ans.* $-\frac{a^2}{12}(3\pi+8a)$

67. A vector $\mathbf{B}$ is always normal to a given closed surface S. Show that $\iiint_V \operatorname{curl}\mathbf{B}\,dV = \mathbf{0}$, where V is the region bounded by S.

68. If $\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{H}\cdot d\mathbf{S}$, where S is any surface bounded by the curve C, show that $\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$.

69. Prove $\oint_C \phi\,d\mathbf{r} = \iint_S d\mathbf{S}\times\nabla\phi$.

70. Use the operator equivalence of Solved Problem 25 to arrive at (*a*) $\nabla\phi$, (*b*) $\nabla\cdot\mathbf{A}$, (*c*) $\nabla\times\mathbf{A}$ in rectangular coordinates.

71. Prove $\iiint_V \nabla\phi\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV$.

72. Let $\mathbf{r}$ be the position vector of any point relative to an origin O. Suppose ϕ has continuous derivatives of order two, at least, and let S be a closed surface bounding a volume V. Denote ϕ at O by ϕ_o. Show that

$$\iint_S \left[\frac{1}{r}\nabla\phi - \phi\nabla\left(\frac{1}{r}\right)\right]\cdot d\mathbf{S} = \iiint_V \frac{\nabla^2\phi}{r}\,dV + \alpha$$

where $\alpha=0$ or $4\pi\phi_o$ according as O is outside or inside S.

73. The potential $\phi(P)$ at a point $P(x,y,z)$ due to a system of charges (or masses) $q_1, q_2, \ldots, q_n$ having position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ with respect to P is given by

$$\phi = \sum_{m=1}^{n}\frac{q_m}{r_m}$$

Prove *Gauss' law*

$$\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi Q$$

where $\mathbf{E} = -\nabla\phi$ is the electric field intensity, S is a surface enclosing all the charges and $Q = \sum_{m=1}^{n} q_m$ is the total charge within S.

74. If a region V bounded by a surface S has a continuous charge (or mass) distribution of density ρ, the potential $\phi(P)$ at a point P is defined by $\phi = \iiint_V \frac{\rho\,dV}{r}$. Deduce the following under suitable assumptions:

(*a*) $\iint_S \mathbf{E}\cdot d\mathbf{S} = 4\pi\iiint_V \rho\,dV$, where $\mathbf{E} = -\nabla\phi$.

(*b*) $\nabla^2\phi = -4\pi\rho$ (Poisson's equation) at all points P where charges exist, and $\nabla^2\phi = 0$ (Laplace's equation) where no charges exist.