Set 10 Solutions

Math 312

1. ~~Change independent varia~~

(a) Try substitution

$$w(z) = z^{\alpha}\phi(z)$$

$$w' = \alpha z^{\alpha-1}\phi + z^{\alpha}\phi'$$

$$w'' = \alpha(\alpha-1)z^{\alpha-2}\phi + 2\alpha z^{\alpha-1}\phi' + z^{\alpha}\phi''$$

Substitute into d.e.:

$$z^2 w'' = \alpha(\alpha-1)z^{\alpha}\phi + 2\alpha z^{\alpha+1}\phi' + z^{\alpha+2}\phi''$$

$$+(1-2\alpha)zw' = \alpha(1-2\alpha)z^{\alpha}\phi + (1-2\alpha)z^{\alpha+1}\phi'$$

$$+(\beta^2\gamma^2 z^{2\gamma} + \alpha^2 - \nu^2\gamma^2)w = \beta^2\gamma^2 z^{2\gamma+\alpha}\phi + (\alpha^2-\nu^2\gamma^2)z^{\alpha}\phi$$

$$\Rightarrow z^{\alpha+2}\phi'' + [(1-2\alpha)+2\alpha]z^{\alpha+1}\phi'$$

$$\alpha^2-\alpha + \alpha^2 - \nu^2\gamma^2 + \alpha - 2\alpha^2$$

$$+[\alpha(\alpha-1) + \alpha^2 - \nu^2\gamma^2 + \alpha(1-2\alpha)]z^{\alpha}\phi$$

$$+\beta^2\gamma^2 z^{2\gamma+\alpha}\phi = 0$$

$$\Rightarrow \phi'' + \frac{1}{z}\phi' + \{\beta^2\gamma^2 z^{2\gamma-2} - \nu^2\gamma^2 z^{-2}\}\phi = 0$$

(b) Change independent variable:

let $t = \beta z^{\gamma} \Rightarrow z = (t/\beta)^{1/\gamma}$

$$\frac{d\phi}{dz} = \frac{dt}{dz}\frac{d\phi}{dt} = \beta\gamma z^{\gamma-1}\frac{d\phi}{dt} \quad \cdots$$ (Chain rule ad nauseum)

$$\frac{d^2\phi}{dz^2} = \frac{d}{dz}\left(\beta\gamma z^{\gamma-1}\frac{d\phi}{dt}\right) =$$

$$= \frac{d}{dz}\left(\beta\gamma z^{\gamma-1}\right)\frac{d\phi}{dt} + \beta\gamma z^{\gamma-1}\frac{dt}{dz}\frac{d^2\phi}{dt^2}$$

$$\left(\frac{d}{dz}\frac{d\phi}{dt} = \frac{dt}{dz}\frac{d}{dt}\frac{d\phi}{dt} = \frac{dt}{dz}\frac{d^2\phi}{dt^2}\right)$$

$$= \beta\gamma(\gamma-1)z^{\gamma-2}\frac{d\phi}{dt} + \beta^2\gamma^2 z^{2\gamma-2}\frac{d^2\phi}{dt^2}$$

Plug into equation:

$$\frac{d^2}{dz^2}\phi + \frac{1}{z}\frac{d}{dz}\phi + \left\{\beta^2\gamma^2 z^{2\gamma} - \nu^2\gamma^2 z^{-2}\right\}\phi = 0$$

$$\Downarrow$$

$$\beta^2\gamma^2 z^{2\gamma-2}\frac{d^2\phi}{dt^2} + \beta\gamma(\gamma-1)z^{\gamma-2}\frac{d\phi}{dt}$$

Combine these

$$+ \frac{1}{z}\cdot\beta\gamma z^{\gamma-1}\frac{d\phi}{dt}$$

$$+\left(\beta^2\gamma^2 z^{2\gamma} - \nu^2\gamma^2 z^{-2}\right)\phi = 0$$

$$\Rightarrow \beta^2\gamma^2 z^{2\gamma-2}\frac{d^2\phi}{dt^2} + \beta\gamma^2 z^{\gamma-2}\frac{d\phi}{dt} + \left(\beta^2\gamma^2 z^{2\gamma} - \nu^2\gamma^2 z^{-2}\right)\phi = 0$$

$\Rightarrow$ (divide through by $\beta^2\gamma^2 z^{2\gamma-2}$:)

$$\frac{d^2\phi}{dt^2} + \frac{1}{\beta z^\gamma}\frac{d\phi}{dz} + \left(1 - \frac{\nu^2}{\beta^2} z^{-2\gamma}\right)\phi = 0$$

But $\beta z^\gamma = t$, $\beta^2 z^{2\gamma} = t^2$,

$$\Rightarrow \quad \frac{d^2\phi}{dt^2} + \frac{1}{t}\frac{d\phi}{dt} + \left(1 - \frac{\nu^2}{t^2}\right)\phi = 0$$

$$\therefore \quad \phi(z(t)) = A J_\nu(t) + B Y_\nu(t)$$
$$= A J_\nu(\beta z^\gamma) + B Y_\nu(\beta z^\gamma)$$

$$\boxed{w(z) = z^\alpha \phi(z) = A z^\alpha J_\nu(\beta z^\gamma) + B z^\alpha Y_\nu(\beta z^\gamma)}$$

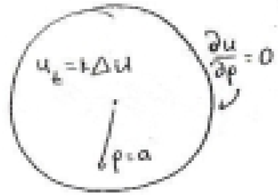
(true also if ν not an integer $\rightarrow$ see problem 5).

(i) $w'' - zw = 0 \Rightarrow z^2 w'' - z^3 w = 0$

Here $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$, $\beta^2\gamma^2 = -1$

$\alpha^2 - \nu^2\gamma^2 = 0$

7.10.3c



Separable variables

$u_t = k\Delta u \quad ; \quad u = q(t)\, W(r,\theta,\phi)$:

$$\frac{q'}{kq} = \frac{\Delta W}{W} = -\lambda \Rightarrow \begin{cases} q'(t) = -\lambda k q \Rightarrow q \sim e^{-\lambda k t} \\ \Delta W + \lambda W = 0 \end{cases}$$

$W = f(\rho)\, g(\theta)\, h(\phi)$

$$\frac{1}{\rho^2}\partial_\rho\left(\rho^2 \frac{\partial f}{\partial \rho}\right)\Big/ f + \frac{1}{\rho^2 \sin\phi}\partial_\phi\left(\sin\phi\, h(\phi)\right)\Big/ h + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 g}{\partial \theta^2}\Big/ g$$
$$+ \lambda = 0$$

$$\Rightarrow \partial_\rho\left(\rho^2 \frac{\partial f}{\partial \rho}\right)\Big/ f + \lambda \rho^2 = \mu = -\left(\frac{1}{\sin\phi}\partial_\phi(\sin\phi\, h)\Big/ h + \frac{1}{\sin^2\phi}\frac{\partial^2 g}{\partial\theta^2}\Big/ g\right)$$

$$\Rightarrow \begin{cases} \frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + (\lambda\rho^2 - \mu) f = 0 \\ \sin\phi \frac{\partial}{\partial\phi}\left(\sin\phi\, h(\phi)\right)\Big/ h + \mu \sin^2\phi = -\frac{\partial^2 g}{\partial \theta^2}\Big/ g = m^2 \Rightarrow \end{cases}$$

$$\Rightarrow \begin{cases} \frac{d^2 g}{d\theta^2} + m^2 g = 0 \\ \frac{d}{d\phi}\left(\sin\phi \frac{dh}{d\phi}\right) + \left(\mu \sin\phi - \frac{m^2}{\sin\phi}\right) h = 0 \end{cases}$$

$\rightarrow$ ($\mu = n(n+1)$ for fns. bounded at $\phi = 0, \pi$)

Finally, combining, the product solution is

$$u(r,\vartheta,\phi,t) = \frac{1}{\rho^{1/2}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{k=1}^{\infty} e^{-\lambda_{n,k} k t} J_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho) P_n^m(\cos\phi) *$$
$$* \left(c_{m,n,k} \cos m\vartheta + d_{m,n,k} \sin m\vartheta \right)$$

The eigenvalues $\lambda_{n,k}$ are determined by demanding

$$\frac{d}{d\rho}\left(\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda}\,\rho)\right)\Big|_{\rho=a} = 0$$

$$\Rightarrow -\tfrac{1}{2} a^{-3/2} J_{n+1/2}(\sqrt{\lambda}\,a) + a^{-1/2}\sqrt{\lambda}\, J'_{n+1/2}(\sqrt{\lambda}\,a) = 0$$

We write $\lambda_{n,k}$ for the k-th root of this.

From I.C.: $\quad u(\rho,\vartheta,\phi,0) = G(r,\phi)\cos 3\vartheta$

$$u(r,\vartheta,\phi,t) = \frac{1}{\rho^{1/2}} \sum_{n=3}^{\infty} \sum_{k=1}^{\infty} c_{n,k}\, e^{-\lambda_{n,k} k t} J_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho) P_n^3(\cos\phi) \cos 3\vartheta$$

$$C_n(\rho) = \frac{\int_{-1}^{1} G(\rho,\phi)\, P_n^3(\cos\phi) \sin\phi \, d\phi}{\int_{-1}^{1} P_n^3(\cos\phi) \sin\phi\, d\phi}$$

$$C_{n,k} = \frac{\int_0^a C_n(\rho)\left(\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho)\right)\rho^2\, d\rho}{\int_0^a J^2_{n+1/2}(\sqrt{\lambda_{n,k}}\,\rho)\, \rho\, d\rho}$$

$\underline{7.10.9c}$ $\quad \underset{\rho<a}{\Delta u = 0}$; $\frac{\partial u}{\partial \rho}(a,\vartheta,\phi) = F(\phi)\cos 4\vartheta$

Using 7.10.28

$$u = \rho^n \begin{pmatrix} \cos m\vartheta \\ \sin m\vartheta \end{pmatrix} P_n^m(\cos\phi)$$

The most general solution:

$$u = \sum_{m=0}^{\infty}{}' \sum_{n=m}^{\infty}{}' \rho^n \left(A_{n,m}\cos m\vartheta + B_{n,m}\sin m\vartheta\right) P_n^m(\cos\phi)$$

Red must satisfy:

$$\frac{\partial u}{\partial \rho}(\rho,\vartheta,\phi)\Big|_{\rho=a} = F(\phi)\cos 4\vartheta = \sum_{m=0}^{\infty}{}' \sum_{n=m}^{\infty} n a^{n-1}\left(A_{n,m}\cos m\vartheta + B_{n,m}\sin m\vartheta\right) P_n^m(\cos\phi)$$

$$= \sum_{n=4}^{\infty} n a^{n-1} A_{n,4} P_n^4(\cos\phi)\cos 4\vartheta$$

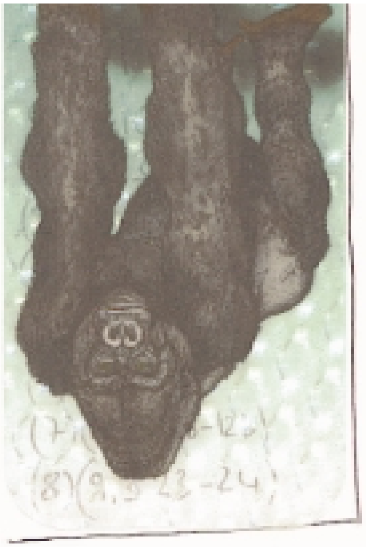
so we need $\quad \sum_{n=4}^{\infty}{}' \left(n a^{n-1} A_{n,4}\right) P_n^4(\cos\phi) = F(\phi)$

$$\sum_{n=4}^{\infty} f_n^{(4)} P_n^4(\cos\phi) =$$

$$n a^{n-1} A_{n,4} = f_n^{(4)} = \frac{\int_{-1}^{1} P_n^4(\cos\phi) F(\phi)\sin\phi\, d\phi}{\int_{-1}^{1} \left(P_n^4(\cos\phi)\right)^2 \sin\phi\, d\phi}$$

$n = 4, 5, 6, \ldots$

$$\boxed{A_{n,4} = f_n^{(4)} / n a^{n-1}}$$



7.10.10c $\quad r>a \quad \Delta\phi=0$

(6/18)

$\phi=+V$ $\quad \Delta\phi=0$

$\phi=-V$

The solution on the exterior resembles the solution on the interior of the sphere (eq. 7.10.3

except for the r-dependence:

$$u(r,\theta,\phi)=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} r^{-n-1}\left(A_{mn}\cos m\theta+B_{mn}\sin m\theta\right)P_n^m(\cos\phi$$

But since the boundary conditions are rotationally symmetric about the ~~θ~~, ϕ-axis, there can be no θ-dependence: $m=0$,

$$u=u(r,\phi)=\sum_{n=0}^{\infty} r^{-n-1}A_{0n}P_n(\cos\phi) \qquad \text{(like 7.10.3}$$

The boundary condition becomes: (like 7.10.34)

$$\sum_{n=0}^{\infty}A_n a^{-n-1}P_n(\cos\phi)=\begin{cases}V, & 0<\phi<\pi/2\\ -V, & \pi/2<\phi<\pi\end{cases}$$

$$\Rightarrow\quad A_n a^{-n-1}=\frac{\int_{-1}^{0}(-V)P_n(x)\,dx+\int_0^1(V)P_n(x)\,dx}{\int_{-1}^{1}P_n^2(x)\,dx}$$

$$=\begin{cases}0, & n \text{ even}\\ 2V\int_0^1 P_n(x)\,dx\Big/\int_{-1}^{1}P_n^2(x)\,dx & \end{cases}$$

$\boxed{7.8.7}$ Consider Bessel DE

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + (z^2 - m^2) f = 0$$

Let $f = y/z^{1/2}$. Show that

$$\frac{d^2y}{dz^2} + y\left(1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right) = 0$$

$$\frac{df}{dz} = z^{-1/2}\frac{dy}{dz} - \frac{1}{2}z^{-3/2}y$$

$$\frac{d^2f}{dz^2} = z^{-1/2}\frac{d^2y}{dz^2} - z^{-3/2}\frac{dy}{dz} + \frac{3}{4}z^{-5/2}y$$

$$\therefore (z^2 - m^2) f = (z^2 - m^2) z^{-1/2} y$$

$$z\frac{df}{dz} = z^{1/2}\frac{dy}{dz} - \frac{1}{2}z^{-1/2}y$$

$$z^2\frac{d^2f}{dz^2} = z^{3/2}\frac{d^2y}{dz^2} - z^{1/2}\frac{dy}{dz} + \frac{3}{4}z^{-1/2}y$$

adding, ~~dividing by z^2:~~

$$\cancel{\frac{d^2y}{dz^2}} \quad z^{3/2}\frac{d^2y}{dz^2} - \cancel{z^{1/2}\frac{dy}{dz}} + \frac{3}{4}z^{-1/2}y$$
$$+ \cancel{z^{1/2}\frac{dy}{dz}} - \frac{1}{2}z^{-1/2}y$$
$$+ z^{3/2}y - m^2 z^{-1/2}y = 0$$

$$\Rightarrow z^{-3/2}\frac{d^2y}{dz^2} + y\left\{1 + \frac{1}{4}z^{-2} - m^2 z^{-2}\right\} = 0$$

(7/18)

7.8.10 Use 7.8.7 to improve on the asymptotic formulas:

$$\begin{cases} J_m(z) \sim \sqrt{\dfrac{2}{\pi z}} \cos \left(z - \dfrac{\pi}{4} - \dfrac{m\pi}{2}\right), \\ Y_m(z) \sim \sqrt{\dfrac{2}{\pi z}} \sin \left(z - \dfrac{\pi}{4} - \dfrac{m\pi}{2}\right) \end{cases} \text{ as } z \to \infty$$

(a) Substitute $y = e^{iz} w(z)$ and show that

$$\frac{d^2 w}{dz^2} + 2i \frac{dw}{dz} + \frac{\gamma}{z^2} w = 0 \quad \text{, with } \gamma = \frac{1}{4} - m^2$$

$$\frac{dy}{dz} = e^{iz} \frac{dw}{dz} + i e^{iz} w$$

$$\frac{d^2 y}{dz^2} = e^{iz} \frac{d^2 w}{dz^2} + 2i e^{iz} \frac{dw}{dz} - e^{iz} w$$

$$\text{so: } e^{iz} \left\{ \frac{d^2 w}{dz^2} + 2i \frac{dw}{dz} - \cancel{w} + w \left\{ \cancel{1} + \frac{1-4m^2}{4z^2} \right\} \right\} = 0$$

$$\Rightarrow \frac{d^2 w}{dz^2} + 2i \frac{dw}{dz} + \left(\frac{1-4m^2}{4z^2}\right) w = 0$$

(b) Substitute $w = \sum_{n=0}^{\infty}{}' \beta_n z^{-n}$. Assuming $\beta_0 = 1$, determine the first few β_n

$$\frac{dw}{dz} = \sum_{n=0}^{\infty} -n \beta_n z^{-n-1}$$

$$\frac{d^2 w}{dz^2} = \sum_{n=0}^{\infty}{}' n(n+1) \beta_n z^{-n-2}$$

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Then:

$$\sum_{n=0}^{\infty} n(n+1) f_n z^{-(n+2)} + 2i \sum_{n=0}^{\infty} -n f_n z^{-(n+1)} + \gamma \sum_{n=0}^{\infty} f_n z^{-(n+2)} = 0$$

$n = k+1$, $k = n-1$ $\Rightarrow n+1 = k+2$

$$= \sum_{k=-1}^{\infty} -(k+1) f_{k+1} z^{-(k+2)}$$

$\longrightarrow$ change to $k=0$, since for $k=-1$ no contribution

$$\Rightarrow \sum_{n=0}^{\infty} \left\{ \underbrace{n(n+1) f_n - (n+1) f_{n+1} \cdot 2i + \gamma f_n}_{\equiv 0 \ \forall n} \right\} z^{-(n+2)} = 0$$

$\Rightarrow n=0$: $\quad -f_1 \cdot 2i + \gamma f_0 = 0 \Rightarrow f_1 = -\frac{\gamma}{2} i \quad (f_0 = 1)$

$n=1$: $\quad 2f_1 - 4i f_2 + \gamma f_1 = 0 \Rightarrow f_2 = \frac{2+\gamma}{4i} f_1$

$$\Rightarrow f_2 = \frac{2+\gamma}{4i} \cdot \left(-\frac{\gamma}{2} i\right) = -\frac{(2+\gamma)\gamma}{\underset{= 2 \cdot 4}{8}}$$

$\vdots$

n: $$\boxed{f_{n+1} = \frac{n(n+1)+\gamma}{2(n+1)i} f_n = -i \frac{n(n+1)+\gamma}{2(n+1)} f_n}$$

$$f_3 = -i \frac{6+\gamma}{6} f_2 = i \frac{\gamma(2+\gamma)(6+\gamma)}{\underset{= 2 \cdot 4 \cdot 6}{48}}$$

$$f_4 = -i \frac{12+\gamma}{8} f_3 = \frac{\gamma(2+\gamma)(6+\gamma)(12+\gamma)}{2 \cdot 4 \cdot 6 \cdot 8 = 2^4 4!} = \frac{\prod_{n=1}^{4} (\gamma + n(n+1))}{2^4 4!}$$

$$f_k = \prod_{n=1}^{k} (\gamma + n(n+1)) \Big/ 2^k k!$$

(c) Use part (b) to obtain an improved asymptotic solution of Bessel's diff. eqn. (For real solutions, take real/imaginary parts).

From (b);

$$w = f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + \cdots$$

$$= 1 - i\frac{\gamma}{2}\frac{1}{z} - \frac{\gamma(\gamma+2)}{2^2 2!}\frac{1}{z^2} + i\frac{\gamma(\gamma+2)(\gamma+6)}{2^3 3!}\frac{1}{z^2} + \frac{\gamma(\gamma+2)(\gamma+6)(\gamma+12)}{2^4\, 4!}\frac{1}{z^4} \cdots$$

$$= \left(1 - \frac{\gamma(\gamma+2)}{8}\frac{1}{z^2} + \frac{\gamma(\gamma+2)(\gamma+6)(\gamma+12)}{2^4 \cdot 4!}\frac{1}{z^4} + \cdots\right) \equiv Re$$

$$- i\left(\frac{\gamma}{2}\frac{1}{z} - \frac{\gamma(\gamma+2)(\gamma+6)}{2^3\, 3!}\frac{1}{z^3} + \cdots\right) \equiv Im$$

Then $f = \frac{1}{\sqrt{z}}y = \frac{e^{iz}}{\sqrt{z}}w$

$$= \frac{1}{\sqrt{z}}(\cos z + i\sin z)\left\{Re + i\,Im\right\}$$

$$= \frac{1}{\sqrt{z}}\left(Re \cdot \cos z - Im\sin z\right) + i\frac{1}{\sqrt{z}}\left(Re\sin z + Im\cos z\right)$$

$$=: \Phi_1 + i\,\Phi_2$$

Now $(\mathrm{Re}\cos z - \mathrm{Im}\sin z)/\sqrt{z}$

$$= \frac{1}{\sqrt{z}}\left(1 - \frac{\gamma(\gamma+2)}{8}\frac{1}{z^2} + \dots\right)\cos z + \frac{1}{\sqrt{z}}\left(\frac{\gamma}{2}\frac{1}{z} - \frac{\gamma(\gamma+2)(\gamma+6)}{48}\frac{1}{z^3}\right)\sin z$$

$$\Phi_1 := \frac{1}{\sqrt{z}}\cos z + \frac{\gamma}{2}\frac{1}{z^{3/2}}\sin z - \frac{\gamma(\gamma+2)}{8}\frac{1}{z^{5/2}}\cos z + \cdots$$

while $(\mathrm{Im}\cos z + \mathrm{Re}\sin z)/\sqrt{z}$

$$= \frac{1}{\sqrt{z}}\left(1 - \frac{\gamma(\gamma+2)}{8}\frac{1}{z^2} + \dots\right)\sin z - \frac{1}{\sqrt{z}}\left(\frac{\gamma}{2}\frac{1}{z} - \frac{\gamma(\gamma+2)(\gamma+6)}{48}\frac{1}{z^3}\right)\cos z$$

$$\Phi_2 := \frac{1}{\sqrt{z}}\sin z - \frac{\gamma}{2}\frac{1}{z^{3/2}}\cos z - \frac{\gamma(\gamma+2)}{8}\frac{1}{z^{5/2}}\sin z + \cdots$$

Both of these expressions are solutions of the DE.

Now, we can combine Φ_1 and Φ_2 to match J or Y to leading order:

(a) $\underline{J_n}$: $\sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{z}}\cos\left(z - \frac{\pi}{4} - m\frac{\pi}{2}\right)$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\left[\cos z\cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right) + \sin z\sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right)\right]$$

i.e. $J_n \approx \sqrt{\frac{2}{\pi}}\left\{\cos\left(\frac{\pi}{4} + \frac{m\pi}{2}\right)\Phi_1 + \sin\left(\frac{\pi}{4} + \frac{m\pi}{2}\right)\Phi_2\right\}$

$$\Rightarrow J_n(z) \approx \frac{1}{\sqrt{z}}\sqrt{\frac{2}{\pi}}\left\{\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\left(1-\frac{\gamma(\gamma+2)}{8}\frac{1}{z^2}+\cdots\right)+\sin\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\left(\frac{\gamma}{2}\frac{1}{z}-\frac{\gamma(\gamma+2)(\gamma+6)}{48}\frac{1}{z^3}\cdots\right)\right\}$$

$$J_n(z) \approx \frac{1}{\sqrt{z}}\sqrt{\frac{2}{\pi}}\left\{\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\left[\cos z+\frac{\gamma}{2}\frac{1}{z}\sin z-\frac{\gamma(\gamma+2)}{8}\frac{1}{z^2}\cos z\cdots\right] + \sin\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\left[\sin z-\frac{\gamma}{2}\frac{1}{z}\cos z-\frac{\gamma(\gamma+2)}{8}\frac{1}{z^2}\sin z\cdots\right]\right\}$$

$$= \frac{1}{\sqrt{z}}\sqrt{\frac{2}{\pi}}\left\{\left[\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\cos z+\sin\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\sin z\right] + \frac{\gamma}{2}\frac{1}{z}\left[\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\sin z-\sin\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\cos z\right]+\cdots\right\}$$

$$= \frac{1}{\sqrt{z}}\sqrt{\frac{2}{\pi}}\left\{\cos\left(z-\frac{\pi}{4}-\frac{m\pi}{2}\right)+\frac{\gamma}{2}\frac{1}{z}\sin\left(z-\frac{\pi}{4}-\frac{m\pi}{2}\right)+O\left(\frac{1}{z^2}\right)\right\}$$

A correction to $Y_n(z)$ is found similarly, by considering

$$Y_n = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\left\{\sin z\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)-\cos z\sin\left(\frac{\pi}{4}+m\frac{\pi}{2}\right)\right\}$$

as the first term in the combination

$$Y_n \sim \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{z}}\left\{\Phi_2\cos\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)+\Phi_1\sin\left(\frac{\pi}{4}+\frac{m\pi}{2}\right)\right\}$$

etc

(d) Recurrence formula for the β_n

That was found in part (b) as

$$\beta_{n+1} = -i \frac{n(n+1)+\gamma}{2(n+1)} \beta_n$$

so that $$\beta_{n+1} = \frac{\prod_{k=1}^{n} (\gamma + k(k+1))}{2^n n!}$$

To show that the series diverges, apply the ratio test:

$$\left| \frac{\beta_{n+1}/z^{n+1}}{\beta_n/z^n} \right| = \frac{n(n+1)+\gamma}{2(n+1)} \cdot \frac{1}{|z|} \to \infty$$

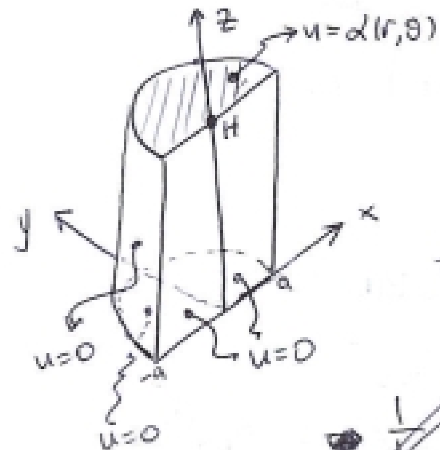
as $n \to \infty$ for any finite $|z|$.

$\Rightarrow$ ~~$|z|>$~~

7.9. 2a Solve Laplace's equation inside a semicircular cylinder subject to the BC.

(a) $u(r,\theta,0)=0$ $\quad u(r,\theta,H)=\alpha(r,\theta)$ $\quad u(r,0,z)=0$

$u(r,\pi,z)=0$ $\quad u(a,\theta,z)=0$



$$\Delta u = 0$$
$$\Downarrow$$
$$u = f(r)\, h(z)\, g(\theta)$$

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)}{f} + \frac{1}{r^2}\frac{d^2g}{d\theta^2}\Big/g + \frac{1}{h}\frac{d^2h}{dz^2} = 0$$

$$\cancel{\frac{1}{h}\frac{d^2h}{dz^2} \Rightarrow}$$

$$\Rightarrow \frac{\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)}{f} + \frac{\frac{1}{r^2}\frac{d^2g}{d\theta^2}}{g} = -\frac{1}{h}\frac{d^2h}{dz^2} = -\lambda \quad , \lambda > 0$$

$$\Rightarrow \frac{d^2h}{dz^2} - \lambda h = 0 \Rightarrow h(z) = \sinh\sqrt{\lambda}\, z$$

(vanishes at $z=0$; but not at $z=H$)

Then $r^2 \frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{d^2g}{d\theta^2} = -\lambda r^2 \Rightarrow$

$$\frac{1}{f} r \frac{d}{dr}\left(r\frac{df}{dr}\right) + \lambda r^2 = -\frac{1}{g}\frac{d^2g}{d\theta^2} = \mu$$

$$\Rightarrow \frac{d^2g}{d\theta^2} + \mu g = 0 \Rightarrow g(\theta) = \cancel{A\cos\theta +} A\cos\sqrt{\mu}\,\theta + B\sin\sqrt{\mu}\,\theta$$

Now $g(0) = 0 \Rightarrow A = 0$; $g(\pi) = B\sin\sqrt{\mu}\,\pi = 0$

$$\Rightarrow \sqrt{\mu} = n \Rightarrow \underline{\mu = n^2}$$

so $g_n(\theta) = \sin n\theta$

Finally $\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r - \frac{n^2}{r}\right)f = 0$

$\Rightarrow f(r) = J_n(\sqrt{\lambda}\, r)$

with $J_n(\sqrt{\lambda}\, a) = 0 \Rightarrow \sqrt{\lambda} = \frac{z_{nm}}{a} \Rightarrow \lambda = \left(\frac{z_{nm}}{a}\right)^2.$

Combining:

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_n(\sqrt{\lambda_{nm}}\, r)\cos n\theta \sinh(\sqrt{\lambda_{nm}}\, z)$$

BC: $$u(r,\theta,H) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_n(\sqrt{\lambda_{nm}}\, r)\cos n\theta \sinh \sqrt{\lambda_{nm}}\, H$$

$$= \alpha(r,\theta)$$

$$\Rightarrow A_{nm}\sinh\sqrt{\lambda_{nm}}\, H = \frac{\int_{r=0}^{a}\int_{\theta=0}^{\pi} \alpha(r,\theta)\sin n\theta\, J_n(\sqrt{\lambda_{nm}}\, r)\, r\, dr\, d\theta}{\left(\iint \sin^2 n\theta\, J_n^2(\sqrt{\lambda_{nm}}\, r)\, r\, dr\, d\theta\right)}$$

$$\left(\iint \sin^2 n\theta\, J_n^2(\sqrt{\lambda_{nm}}\, r)\, r\, dr\, d\theta\right) = N_{nm} \equiv \frac{\pi}{2}\cdot\frac{a^2}{2}\left(J_n'(\sqrt{\lambda_{nm}}\, a)\right)^2$$

$$A_{nm} = \frac{1}{N_{nm}\sinh\sqrt{\lambda_{nm}}\, H}\int_{r=0}^{a}\int_{\theta=0}^{\pi} \alpha \sin n\theta\, J_n(\sqrt{\lambda_{nm}}\, r)\, r\, dr\, d\theta$$

(16/18)

7.9.3c Solve the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

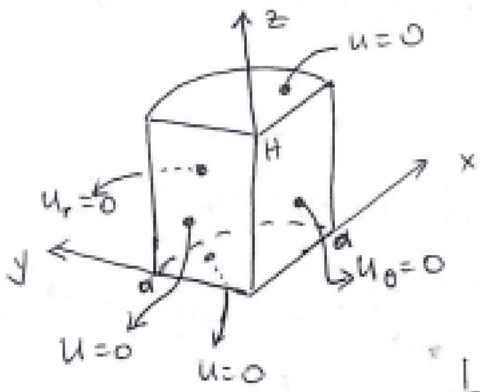
inside a quarter-circular cylinder ($0 < \theta < \frac{\pi}{2}$, with radius a and height H) subject to the I.C.

$$u(r,\theta,z,0) = f(r,\theta,z)$$

and discuss the limit $t \to \infty$ for the BC

$u(r,\theta,0) = 0$ $\quad u(r,\theta,H) = 0$ $\quad \frac{\partial u}{\partial \theta}(r,0,z) = 0$

$u(r,\frac{\pi}{2},z) = 0$ $\quad \frac{\partial u}{\partial r}(a,\theta,z) = 0$



Now

$u = f(r)g(\theta)h(z)\phi(t)$

First separate the time:

$u = V(r,\theta,z)\,\phi(t)$:

$$\frac{1}{k\phi}\frac{d\phi}{dt} = \frac{1}{V}\nabla^2 V = -\sigma$$

$$\Rightarrow \phi(t) = e^{-k\sigma t}; \quad \nabla^2 V + \sigma V = 0$$

(Helmholtz eq.)

σ to be determined.

Now $\frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{1}{r^2}\frac{d^2g}{d\theta^2} + \frac{1}{h}\frac{d^2h}{dz^2} + \sigma = 0$

$\Rightarrow \frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{1}{r^2}\frac{d^2g}{d\theta^2} + \sigma = -\frac{1}{h}\frac{d^2h}{dz^2} = \lambda$

$\Rightarrow \frac{d^2h}{dz^2} + \lambda h = 0$ with $h(H) = 0,\ h(0) = 0$

$\Rightarrow h(z) = \sin\sqrt{\lambda}\, z$

$\sin\sqrt{\lambda} H = 0 \Rightarrow \sqrt{\lambda} H = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{H}\right)^2$

So $\frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{1}{r^2}\frac{d^2g}{d\theta^2} + \sigma = \left(\frac{n\pi}{H}\right)^2$

$\Rightarrow \frac{1}{f} r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left[\sigma - \left(\frac{n\pi}{H}\right)^2\right] r^2 = -\frac{1}{g}\frac{d^2g}{d\theta^2} = k$

with $\frac{d^2g}{d\theta^2} + kg = 0 \Rightarrow g = A\cos\sqrt{k}\theta + B\sin\sqrt{k}\theta$

$\frac{dg}{d\theta}(0) = 0,\ g\left(\frac{\pi}{2}\right) = 0$: $\frac{dg}{d\theta} = -\sqrt{k}\left(A\sin\sqrt{k}\theta - B\cos\sqrt{k}\theta\right)$

$\Rightarrow B = 0$

i.e. $g(\theta) = A\cos\sqrt{k}\theta$

$g\left(\frac{\pi}{2}\right) = A\cos\sqrt{k}\frac{\pi}{2} = 0 \Rightarrow \sqrt{k}\frac{\pi}{2} = m\pi + \frac{\pi}{2}$

$\sqrt{k} = (2m+1) \Rightarrow k = (2m+1)^2$

So far: $h_n(z) = \sin\frac{n\pi z}{H}$;

$g_m(\theta) = \cos(2m+1)\theta$; $k = (2m+1)^2$

$$\Rightarrow \frac{1}{f} r \frac{d}{dr}\left(r\frac{df}{dr}\right) + \left[\sigma - \left(\frac{n\pi}{H}\right)^2\right] r^2 = (2m+1)^2$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{df}{dr}\right) + \left\{\left(\sigma - \left(\frac{n\pi}{H}\right)^2\right) r - \frac{(2m+1)^2}{r}\right\} f = 0$$

Since we want the solution of this equation to vanish at $r=a$, we must get J (not I!)

so we need $\sigma - \left(\frac{n\pi}{H}\right)^2 = \lambda > 0$ and

$f(r) = J_{2m+1}(\sqrt{\lambda}\, r)$ with $\sqrt{\lambda}\, a = z_{2m+1,\ell}$

$$\Rightarrow \lambda_{m,\ell} = \left(\frac{z_{2m+1,\ell}}{a}\right)^2 \quad \text{(the } \ell\text{-th zero of } J_{2m+1}\text{)}$$

$$\Rightarrow \sigma_{m,n} = \left(\frac{z_{2m+1,\ell}}{a}\right)^2 + \left(\frac{n\pi}{H}\right)^2$$

Finally:

$$u(r,\theta,z,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{\ell mn}\, e^{-\sigma_{m,n} k t} \sin\frac{n\pi z}{H} \cos(2m+1)\theta\, J_{2m+1}(\sqrt{\lambda_{m\ell}}\, r)$$

with $f(r,\theta,z) = \sum\sum\sum A_{\ell mn} \sin\frac{n\pi z}{H} \cos(2m+1)\theta\, J_{2m+1}(\sqrt{\lambda_{m\ell}}\, r)$

$$A_{\ell mn} = \frac{1}{N_{\ell mn}} \iiint f \sin\frac{n\pi z}{H} \cos(2m+1)\theta\, J_{2m+1}(\sqrt{\lambda}\, r) \cdot r\,dr\,d\theta\,dz$$