Math. 312
Set II

2.3.8, 2.4.1(a,b,c)
2.4.2, 2.5.1(a,e)

2.3.8 $\partial_t u = k\,\partial_{xx} u - \alpha u$

$u(0,t) = u(L,t) = 0$

(a) $\alpha > 0$: equilibrium: $(k>0)$

$$\partial_{xx} u - \left(\frac{\alpha}{k}\right) u = 0 \;,\; u(0) = u(L) = 0$$

$$u = A \sinh\sqrt{\tfrac{\alpha}{k}}\, x + B \cosh\sqrt{\tfrac{\alpha}{k}}\, x$$

$$\left.\begin{array}{l} u(0) = A\cdot 0 + B\cdot 1 = 0 \Rightarrow B = 0 \\ u(L) = A \sinh\sqrt{\tfrac{\alpha}{k}}\, L = 0 \Rightarrow A = 0 \end{array}\right\} u(x) \equiv 0 \text{ only equilibrium}$$

(b) $u = X(x)T(t)$:

$$\frac{XT' = kX''T - \alpha XT}{XT} \Rightarrow \frac{T'}{T} = k\frac{X''}{X} - \alpha$$

$$\Rightarrow \frac{T'}{T} + \alpha = k\frac{X''}{X} = -\lambda \quad (\lambda > 0 \text{ for oscillatory behavior in } x)$$

Then $T' + (\alpha+\lambda)T = 0 \Rightarrow T(t) = C e^{-(\alpha+\lambda)t}$

$$X'' + (\lambda/k) X = 0 \Rightarrow X(x) = A \cos\sqrt{\tfrac{\lambda}{k}}\, x + B \sin\sqrt{\tfrac{\lambda}{k}}\, x$$

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B \sin\sqrt{\tfrac{\lambda}{k}}\, x$$

$$X(L) = 0 \Rightarrow B \sin\sqrt{\tfrac{\lambda}{k}}\, L = 0$$

$$\Rightarrow \sqrt{\tfrac{\lambda}{k}}\, L = \frac{n}{\pi} \Rightarrow \lambda = k\left(\frac{n\pi}{L}\right)^2$$

i.e. $u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\alpha + k(\frac{n\pi}{L})^2)t}$

and $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\alpha + k(\frac{n\pi}{L})^2)t}$

$$\xrightarrow{t\to\infty} 0$$

$$\left(B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx\right)$$

2.4.1 $\quad u_t = k u_{xx} \,,\quad 0<x<L,\ t>0\,,$

$$u_x(0,t) = u_x(L,t) = 0 \quad , \ t>0$$

In text (2.4.19) u is found as

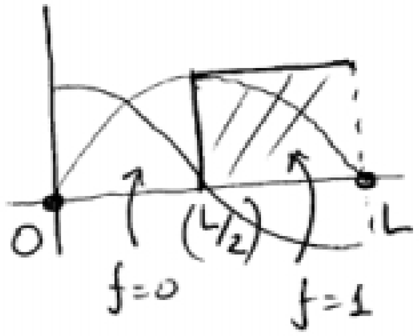
$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-(n\pi/L)^2 kt}$$

with (2.4.23-24)

$$\left.\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(x)\,dx \\ A_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \end{aligned}\right\}$$

Then

(a) $\quad u(x,0) = \begin{cases} 0\,, & x<L/2 \\ 1\,, & x>L/2 \end{cases}$



$$A_0 = \frac{1}{L}\int_{L/2}^{L} dx = \frac{1}{2}$$

$$A_n = \frac{2}{L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{x=L/2}^{L}$$

$$= \frac{2}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left(0 - \begin{cases} 0, & n=2,4,\dots \\ 1, & n=1,5,9,\dots \\ -1, & n=3,7,\dots \end{cases}\right)$$

$$\Rightarrow \begin{cases} A_2 = A_4 \dots = A_{2n} = 0 \\ A_{4n+1} = -\dfrac{2}{(4n+1)\pi} \\ A_{4n+3} = \dfrac{2}{(4n+3)\pi} \end{cases} \quad \text{or} \quad \begin{cases} A_n = -\dfrac{2}{n\pi}\sin\dfrac{n\pi}{2} \\ A_0 = \dfrac{1}{2} \end{cases}$$

$$\left(\text{since} \quad \begin{aligned} \sin\frac{\pi}{2} &= \sin\left(\frac{\pi}{2}+2k\pi\right) = 1 \\ \sin\frac{3\pi}{2} &= \sin\left(\frac{3\pi}{2}+2k\pi\right) = -1 \end{aligned}\right)$$

2.4.1 (c)

$$u(x,0) = -2\sin\frac{\pi x}{L}$$

$$A_0 = \frac{-1}{L}\int_0^L 2\sin\frac{\pi x}{L}\,dx = \frac{2}{\cancel{L}}\cdot\frac{\cancel{L}}{\pi}\cos\frac{\pi x}{L}\Big|_0^L$$

$$= \frac{2}{\pi}\left(\cos\pi - 1\right) = \frac{-4}{\pi}$$

$$A_n = -\frac{2}{L}\int_0^L 2\sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

(Use formula: $2\sin a\cos b = \sin(a+b) + \sin(a-b)$)

$$2\sin\frac{\pi x}{L}\cos\frac{n\pi x}{L} = \sin\left(\frac{\pi(1+n)}{L}x\right) + \sin\left(\frac{(1-n)\pi x}{L}\right)$$

$$= \sin\left[\frac{(n+1)\pi x}{L}\right] - \sin\left[\frac{(n-1)\pi}{L}x\right]$$

$$A_n = -\frac{2}{L}\int_0^L \left\{\sin\left[\frac{(n+1)\pi x}{L}\right] - \sin\left[\frac{(n-1)\pi x}{L}\right]\right\}dx$$

$$= \frac{2}{\cancel{L}}\,\frac{\cancel{L}}{\pi(n+1)}\cos\frac{(n+1)\pi}{L}x\Big|_0^L - \frac{2}{L}\cdot\frac{L}{(n-1)\pi}\cos\frac{(n-1)\pi x}{L}\Big|_0^L \qquad (n \neq 1)$$

$$= \frac{2}{\pi(n+1)}\left(\cos(n+1)\pi - 1\right) - \frac{2}{\pi(n-1)}\left(\cos(n-1)\pi - 1\right)$$

$$= \begin{cases} \dfrac{4}{\pi(n+1)} - \dfrac{4}{\pi(n-1)} = -\dfrac{8}{\pi(n^2-1)}, & n \text{ even} \\[2ex] 0, & n \text{ odd} \end{cases}$$

$$A_1 = -\frac{2}{L}\int_0^L \sin\frac{2\pi x}{L}\,dx = \frac{\cancel{2}}{\cancel{L}}\,\frac{\cancel{L}}{\cancel{2}\pi}\cos\frac{2\pi x}{L}\Big|_0^L = 0$$

2.4.1 d $\quad u(x,0) = -3\cos\frac{8\pi x}{L}$

$$A_0 = -\frac{3}{L}\int_0^L \cos\frac{8\pi x}{L}\,dx = 0$$

$$\vdots$$

$$A_k = 0, \quad k \neq 8 \quad (\text{by } 2.4.22)$$

$$A_8 = -\frac{6}{L}\int_0^L \underbrace{\cos^2\frac{8\pi x}{L}}_{=\frac{1}{2}+\frac{1}{2}\cos\frac{16\pi x}{L}}\,dx = -\frac{3}{L}\int_0^L dx = -3$$

2.4.1 b $\quad u(x,0) = 6 + 4\cos\frac{3\pi x}{L}$

By inspection: $A_0 = 6$, $A_3 = 4$, others $= 0$

2.4.2 $\quad u_t = k u_{xx}, \quad u_x(0,t) = u(L,t) = 0$

$u = XT; \quad XT' = kX''T \Rightarrow$

$$T'/T = kX''/X = -\lambda$$

$$T' = -\lambda T \Rightarrow T(t) = Ce^{-\lambda t}$$

$$X'' + \left(\frac{\lambda}{k}\right)X = 0 \Rightarrow X = A\cos\sqrt{\tfrac{\lambda}{k}}\,x + B\sin\sqrt{\tfrac{\lambda}{k}}\,x$$

$$X'(0) = 0 \Rightarrow -A\sqrt{\tfrac{\lambda}{k}}\sin\sqrt{\tfrac{\lambda}{k}}\,x + B\sqrt{\tfrac{\lambda}{k}}\cos\sqrt{\tfrac{\lambda}{k}}\,x\Big|_{0}$$

$$\Rightarrow B = 0$$

$$X(L) = A\cos\sqrt{\tfrac{\lambda}{k}}\,L = 0$$

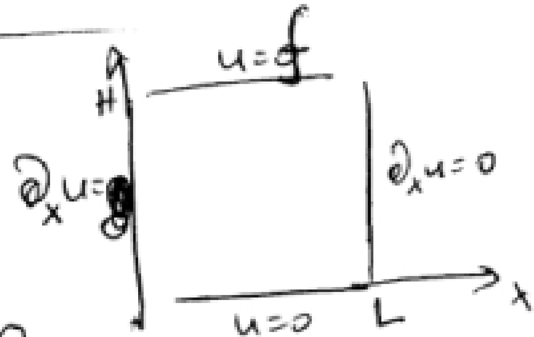
$n = 0, 1, \ldots \quad \Rightarrow \sqrt{\tfrac{\lambda}{k}}\,L = (n+\tfrac{1}{2})\pi \Rightarrow \begin{cases} \lambda = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}k \\ \sqrt{\frac{\lambda}{k}} = (n+\frac{1}{2})\frac{\pi}{L} \end{cases}$

Then

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{(n+\frac{1}{2})\pi x}{L}\, e^{-\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 k t}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos \frac{(n+\frac{1}{2})\pi x}{L}\, dx$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\, dx$$



2.5.1 a $\quad u = XY$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$\Rightarrow \quad X'' + X = 0 \qquad X'(0) = X'(L) = 0$ (oscillatory in x)

$\Rightarrow \quad X_n(x) = A_n \cos \frac{n\pi x}{L} \quad ; \quad \lambda = \left(\frac{n\pi}{L}\right)^2$

$$Y'' - \left(\frac{n\pi}{L}\right)^2 Y = 0 \quad , \; Y(0) = 0$$

$$Y(y) = \sinh \frac{n\pi y}{L} \qquad (Y(0) = 0)$$

$$u = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

$$u(x,H) = \sum_{n=1}^{\infty} \left(A_n \sinh \frac{n\pi H}{L}\right) \cos \frac{n\pi x}{L}$$

$$\Rightarrow \quad \boxed{A_n = \frac{2}{L \sinh \frac{n\pi H}{L}} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx}$$

2.5.1 $u(0,y) = u(L,y) = 0$

$$u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0 \quad , \quad u(x,H) = f$$

Now $\quad X(x) = \cancel{A\cos\frac{n\pi x}{L}} + A\sin\frac{n\pi x}{L}$

$$Y(y) = A\sinh\frac{n\pi y}{L} + B\cosh\frac{n\pi y}{L}$$

$$Y(0) = B, \quad Y'(0) = \frac{n\pi}{L}A$$

$$Y(0) - Y'(0) = B - \frac{n\pi}{L}A = 0 \Rightarrow B = \frac{n\pi}{L}A$$

$$Y(y) = A\left[\sinh\frac{n\pi y}{L} + \frac{n\pi}{L}\cosh\frac{n\pi y}{L}\right]$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n\left[\sinh\frac{n\pi y}{L} + \frac{n\pi}{L}\cosh\frac{n\pi y}{L}\right]\sin\frac{n\pi x}{L}$$

$$A_n = \frac{1}{L\left(\sinh\frac{n\pi H}{L} + \frac{n\pi}{L}\cosh\frac{n\pi H}{L}\right)}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$