Math. 313, Complex Variables
Practice Exam I —- Spring 2004

V. Coutsias

1. (15pts.)

   (a) (5pts) For which complex numbers does
   $$\left|e^{-iz}\right| < 1$$
   hold?

   (b) (10pts) Find all values of
   $$\sqrt{3+4i} + \sqrt{3-4i} \quad .$$

2. (10pts.) Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

3. (20pts.) Find all values of z such that

   (a) $e^z = 1 + i\sqrt{3}$.

   (b) $\tan z = 2i$.

   (c) $z^i = 1$.

   (d) $\log\left(i + \sqrt{z^2+3}\right) = -\frac{\pi}{2}i$.

4. (15pts.) Show that $u(x, y) = \ln\sqrt{x^2+y^2}$ is harmonic in some domain (which?) and find a harmonic conjugate $v(x, y)$.

5. (10pts.) Find a linear fractional transformation that maps the points $\{0, 1, i\}$ to the points $\{1, 0, \infty\}$.

6. (10pts.) Show that
   $$\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$$
   by using the equivalence
   $$\lim_{z\to\infty} f(z) = \infty \Longleftrightarrow \lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0 \; .$$

7. (10pts.) Find the angle of rotation and stretching factor produced by the mapping $w = 1/z^2$ at $z = 1$.

8. (10pts.) Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation $w = z^3$.