

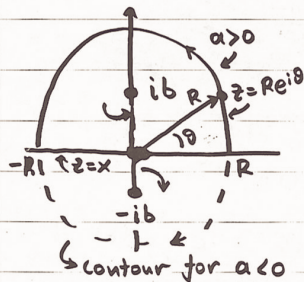
9.1/8

B-V-A: $\frac{\pi}{\sin \pi} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, 0 < a < 1$

METHODS OF C- INTEGRATION

(i) $\int_{-\infty}^{\infty} f(x) dx$; $|f(z)| < \frac{M}{|z|^{1+\epsilon}}$ for $|z| > R_0$
 continuation of $f(z)$ on either upper or lower halfplane

Ex. $I(a) = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2} dx$; $a, b > 0$
 $I = \text{Re } I_1 = \text{Re} \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2} dx = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2} dx$ (even)



$\int_{-R}^R \frac{e^{iax}}{x^2+b^2} dx + \int_0^\pi \frac{e^{ia z} R i e^{i\theta} d\theta}{R^2 e^{2i\theta} + b^2}$
 $I(a) \propto R \rightarrow \infty$
 $= 2\pi i \cdot \text{Res}(ib) = 2\pi i \cdot \frac{e^{ia z}}{z+ib} \Big|_{ib}$
 $= \frac{\pi}{b} e^{-ab}$

9.2/8

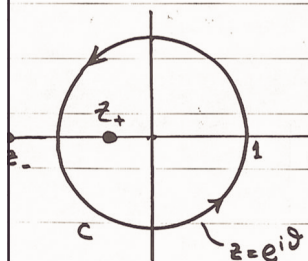
(2) $\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta = \oint_{|z|=1} R(\cos \theta, \sin \theta) \frac{dz}{iz}$

$z = e^{i\theta}$; $dz = ie^{i\theta} d\theta = iz d\theta$; $\frac{1}{2}(z+\frac{1}{z})$; $\frac{1}{2i}(z-\frac{1}{z})$

$I = 2\pi i \sum \text{Res}(\tilde{R}(z))$ where

$\tilde{R}(z) = R(\frac{1}{2}(z+\frac{1}{z}), \frac{1}{2i}(z-\frac{1}{z})) \cdot \frac{1}{z}$

Ex: $I = \int_0^{2\pi} \frac{d\theta}{1+\alpha \cos \theta}$; $|\alpha| < 1$



$I = \frac{1}{i} \oint_{|z|=1} \frac{dz}{z} \frac{1}{1+\frac{\alpha}{2}(z+\frac{1}{z})} = \frac{2}{i} \oint_{|z|=1} \frac{dz}{\alpha z^2 + 2z + \alpha}$

$z_{\pm} = -\frac{1}{\alpha} \pm \sqrt{\frac{1}{\alpha^2} - 1}$; $z_+ z_- = 1$, $\alpha \neq 1$

$z_+ = -\frac{1}{\alpha} + \sqrt{\frac{1}{\alpha^2} - 1}$ only one root inside (z_+)

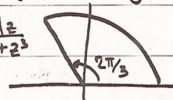
$I = \frac{2}{i} \cdot 2\pi i \cdot \text{Res}(z_+) =$
 $= 4\pi \frac{1}{\alpha(z_+ - z_-)} = \frac{2\pi}{\sqrt{1-\alpha^2}}$

9.3/8

(3) $\int_0^\infty f(x) dx$ where $f(z)$ has symmetry

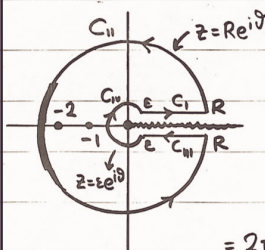
Ex: $\int_0^\infty \frac{dx}{1+x^3} \rightarrow \oint_C \frac{dz}{1+z^3}$

Since $(re^{2\pi i/3})^3 = r^3$



This is not always possible!

Ex. $I = \int_0^\infty \frac{dx}{x^2+3x+2}$



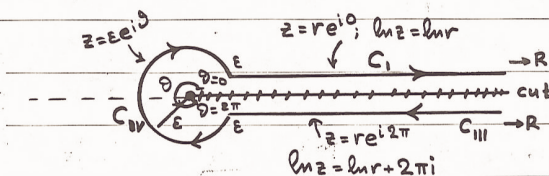
Consider

$B = \oint_C \frac{\ln z dz}{z^2+3z+2}$

Show $C_{II}, C_{IV} \rightarrow 0$

Then $\int_{C_I} - \int_{C_{III}} = 2\pi i \sum \text{Res} = -2\pi i I$ (see next p.)
 $= 2\pi i \left(\frac{\ln(2e^{i\pi})}{-2+1} + \frac{\ln(e^{i\pi})}{-1+2} \right) = -2\pi i \ln 2$
 $\frac{z+1}{z=-2} \quad \frac{z+2}{z=-1} \Rightarrow I = \ln 2$

9.4/8



on $C_I: \frac{\ln z}{z^2+3z+2} = \frac{\ln r + 0i}{r^2+3r+2}$

on $C_{III}: \frac{\ln z}{z^2+3z+2} = \frac{\ln r + 2\pi i}{r^2+3r+2}$

$\int_{C_I} + \int_{C_{III}} = \int_\epsilon^R \frac{\ln r - (\ln r + 2\pi i)}{r^2+3r+2} dr = -2\pi i \int_\epsilon^R \frac{dr}{r^2+3r+2}$
 $\xrightarrow{R \rightarrow \infty} -2\pi i I$
 $\xrightarrow{\epsilon \rightarrow 0} -2\pi i I$

$C_{IV}: z = \epsilon e^{i\theta} : \left| \int_{C_{IV}} \right| \leq \int_{C_{IV}} \frac{|\ln \epsilon + i\theta| |\epsilon e^{i\theta}| d\theta}{|\epsilon^2 e^{2i\theta} + 2\epsilon e^{i\theta} + 3|} \leq \int_0^{2\pi} \frac{(\ln \epsilon + \theta) \epsilon d\theta}{3 - 2\epsilon - \epsilon^2}$
 $\xrightarrow{\epsilon \rightarrow 0} 0$

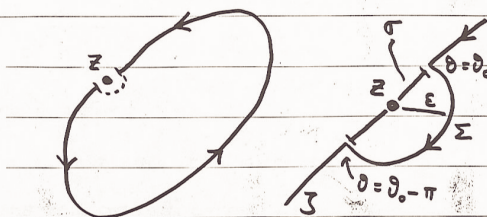
$C_{II}: z = R e^{i\theta} : \left| \int_{C_{II}} \right| \leq \int_{C_{II}} \frac{(\ln R + \theta) R d\theta}{R^2 - 3R - 2} \xrightarrow{R \rightarrow \infty} 0$
 $\epsilon \ln \epsilon \rightarrow 0, \quad \frac{\ln R}{R} \xrightarrow{R \rightarrow \infty} 0$

9.5/8

In general, if $|f(z)| < \frac{M}{R^{1+\delta}}$, $|z| > R_0$,
 then can evaluate
 $\int_0^\infty f(x) dx$ by key-hole contour for
 $\oint_C \ln z f(z) dz$.

Note: above we assumed that $f(z)$ (the
 continuation of $f(x)$ for complex values) had no poles on the $(+)$
 real axis. In fact we can also allow simple poles on the
 real axis. For that we need to introduce the
 "Cauchy Principal Value"

9.6/8



$$I = \oint_C \frac{f(z)}{z-z} dz = \lim_{\epsilon \rightarrow 0} \int_{C-\sigma} \frac{f(z)}{z-z} dz$$

let $C' = C - \sigma$; then

$$\oint_{C'+\Sigma} \frac{f(z)}{z-z} dz = 0 \quad (\text{no poles inside}).$$

Cauchy principal value,
 let "lips" of excluded
 interval close at the
 same rate.

$$\text{And: } \lim_{\epsilon \rightarrow 0} \oint_{C'} \equiv \text{P.V.} \oint_C \quad (\text{desired integral})$$

$$\text{But } \lim_{\epsilon \rightarrow 0} \int_{\Sigma} \frac{f(z)}{z-z} dz = f(z) \lim_{\epsilon \rightarrow 0} \int_{\theta_0}^{\theta_0 - \pi} \frac{iz e^{i\theta} d\theta}{\epsilon e^{i\theta}} = -\pi i f(z)$$

$$z = z + \epsilon e^{i\theta}, \quad \theta_0 - \pi \leq \theta \leq \theta_0 \quad f(z) = f(z + \epsilon e^{i\theta}) \xrightarrow{\epsilon \rightarrow 0} f(z)$$

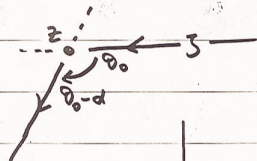
9.7/8

So:
$$\text{PV} \oint \frac{f(z)}{z-z} dz = \pi i f(z)$$

 (instead of $2\pi i f(z)$
 if z was interior)

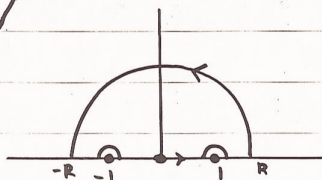
Note: if contour had an angle α at z (i.e. it was not smooth), then

$$\text{PV} \oint \frac{f(z)}{z-z} dz = i\alpha f(z)$$



Ex:
$$\text{PV} \int_{-\infty}^{\infty} \frac{dx}{x^2-1} = \pi i \left(\frac{1}{2} \frac{1}{z+1} \Big|_{z=1} + \frac{1}{2} \frac{1}{z-1} \Big|_{z=-1} \right)$$

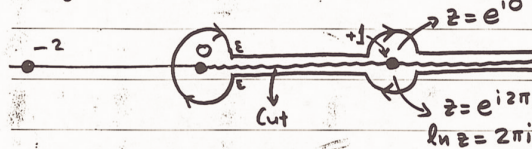
$$= 0$$



9.8/8

Ex: (simple pole on the branch cut)

$$I = \int_0^{\infty} \frac{dz}{z^2+z-2} : \oint \frac{\ln z dz}{z^2+z-2} \quad \ln z = 0$$



Work as before. But now

$$-2\pi i I = 2\pi i \left(\text{Res}(z=-2) + \frac{1}{2} \text{Res}(z=e^{i0}) + \frac{1}{2} \text{Res}(z=e^{i2\pi}) \right)$$

$$\text{Res}(e^{i0}) = \frac{\ln z}{z+2} \Big|_{z=e^{i0}} = \frac{0}{3}$$

$$\text{Res}(e^{i2\pi}) = \frac{\ln z}{z+2} \Big|_{z=e^{i2\pi}} = \frac{2\pi i}{3}$$

$$\text{Res}(-2=2e^{i\pi}) = \frac{\ln z}{z+1} \Big|_{z=2e^{i\pi}} = \frac{\ln 2 + i\pi}{-3}$$

$$I = - \left(\frac{\ln 2 + i\pi}{-3} + 0 + \frac{1}{2} \left(\frac{2\pi i}{3} \right) \right) = \ln 2 / 3$$

10/8

$$(5) I = \int_{-\infty}^{\infty} e^{iax} f(x) dx, a > 0$$

with $|f(z)| < \frac{M}{|z|}$, for $|z| > R_0$.

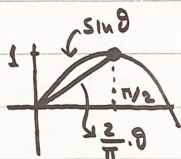
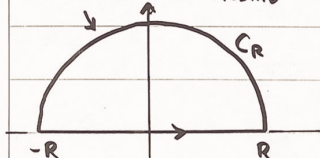
$$I = \lim_{R \rightarrow \infty} \left(\int_{-R}^R + \int_{C_R} \right) = 2\pi i \sum (\text{Res in upper half-plane})$$

(if $a < 0$, complete in lower half).

Jordan lemma: $\left| \int_{C_R} f e^{iax} dz \right| \xrightarrow{R \rightarrow \infty} 0$

$$\begin{aligned} \left| \int_{C_R} \dots \right| &\leq \int_{C_R} |f(Re^{i\theta})| \cdot e^{-aR \sin \theta} R d\theta < M \int_0^\pi e^{-aR \sin \theta} d\theta \\ &= 2M \int_0^{\pi/2} e^{-aR \sin \theta} d\theta \\ &\leq 2M \int_0^{\pi/2} e^{-aR \cdot \frac{2\theta}{\pi}} d\theta = \end{aligned}$$

$$z = Re^{i\theta} = R \cos \theta + iR \sin \theta$$



$$\frac{M\pi}{aR} (1 - e^{-aR}) \xrightarrow{R \rightarrow \infty} 0 \text{ if } a > 0.$$

$$\sin \theta \geq \frac{2}{\pi} \theta, 0 \leq \theta \leq \frac{\pi}{2}$$

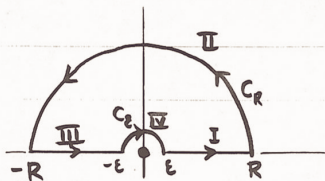
10.2/8

Ex $\int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx, a > 0$ (Principal value)

Consider $\oint \frac{e^{iaz}}{z} dz = PV \int_{-R}^R + \int_{C_R} + \int_{C_\epsilon} = 0$ (no poles inside)

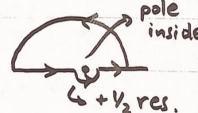
Since $\lim_{R \rightarrow \infty} \int_{C_R} = 0$ by Jordan lemma,

$$PV \int_{-\infty}^{\infty} \frac{e^{iax}}{x} dx = \pi i \text{ Res}(z=0) = \pi i$$



i.e. $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$

(what if we used:



Note: C_ϵ contributes $-1/2$ the residue at $z=0$

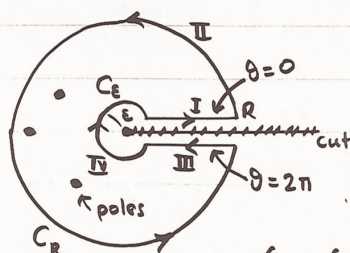
10.3/8

$$(6) I = \int_0^\infty x^{\alpha-1} f(x) dx, \quad 0 < \alpha < 1$$

$|f(z)| \rightarrow 0$ as $|z| \rightarrow \infty$; only poles ("meromorphic").

$$\text{Consider } \oint z^{\alpha-1} f(z) dz = 2\pi i \sum \text{Res}^{(n)} \quad \text{all poles}$$

(if poles on + real axis, take principal values, get $1/2$ residue from each side of branch cut). Can show:



$$(i) \int_{C_I} \dots \rightarrow I \text{ as } R \rightarrow \infty, \epsilon \rightarrow 0$$

$$(ii) \text{ on } C_{III}, z = re^{i2\pi}, z^{\alpha-1} = r^{\alpha-1} e^{i2\pi(\alpha-1)} \\ \int_{III} \rightarrow -e^{i2\pi(\alpha-1)} I$$

while $\int_{C_{II}}, \int_{C_{IV}} \rightarrow 0$ as $R \rightarrow \infty, \epsilon \rightarrow 0$

10.4/8

$$\text{so: } \oint_{III} \rightarrow (1 - e^{i2\pi(\alpha-1)}) I = 2\pi i \sum (\text{Res}).$$

$$\text{Ex. } \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx : \oint \frac{z^{\alpha-1}}{z+1} dz = \\ = 2\pi i \cdot \frac{z^{\alpha-1}}{1} \Big|_{z=e^{i\pi}} = 2\pi i e^{i\pi(\alpha-1)}$$

pole: $z = -1$

$$(I) \int_\epsilon^R \rightarrow I$$

$$(III) \int_R^\epsilon \frac{(re^{i2\pi})^{\alpha-1} e^{i2\pi} dr}{re^{i2\pi} + 1} \Rightarrow -e^{i2\pi(\alpha-1)} I$$

$$\text{on II: } \left| \int_0^{2\pi} \dots \right| \leq \int_0^{2\pi} \frac{|Re^{i\theta}|^{\alpha-1} R d\theta}{|Re^{i\theta} + 1|} \leq \int_0^{2\pi} \frac{R^{\alpha-1} R}{R-1} d\theta = \frac{2\pi R^\alpha}{R-1} \xrightarrow{R \rightarrow \infty} 0 \quad (\alpha < 1)$$

$$\text{on IV: } \left| \int_{2\pi}^0 \dots \right| \leq \int_0^{2\pi} \frac{|\epsilon e^{i\theta}|^{\alpha-1} \epsilon d\theta}{|1 + \epsilon e^{i\theta}|} \leq \frac{\epsilon^\alpha}{1-\epsilon} \cdot 2\pi \xrightarrow{\epsilon \rightarrow 0} 0 \quad (\alpha > 0)$$

$$\text{So: } (1 - e^{i2\pi(\alpha-1)}) I = 2\pi i \cdot e^{i\pi(\alpha-1)} \Rightarrow I = \frac{2\pi i e^{i\pi(\alpha-1)}}{1 - e^{i2\pi(\alpha-1)}}$$

$$\Rightarrow I = \frac{\pi}{\sin \pi(1-\alpha)}$$

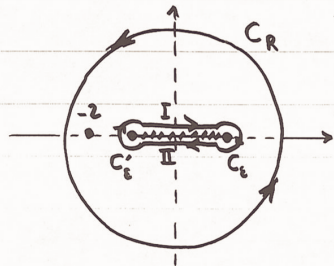
10.5/8

(7) Special problems

(a) $\int_{-1}^1 \frac{f(x) dx}{\sqrt{1-x^2}}$ integrate around cut
($|f| \rightarrow 0, |z| \rightarrow \infty$)

Ex. I: $\int_{-1}^1 \frac{dx}{(x+2)\sqrt{1-x^2}}$

Consider $\oint \frac{dz}{(z+2)\sqrt{1-z^2}} =$



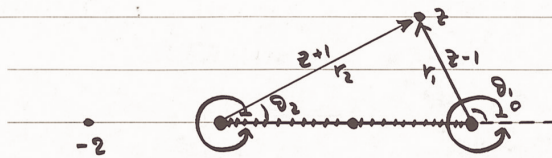
$$= 2\pi i \cdot \text{Res}(-2) = 2\pi i \cdot \frac{1}{\sqrt{1-z^2}} \Big|_{-2}$$

$$= 2\pi i \cdot \frac{1}{\pm i\sqrt{3}} = \pm \frac{2\pi}{\sqrt{3}}$$

Problem: we must define sq. root
so we get (+) value for I!

As $R \rightarrow \infty, \epsilon \rightarrow 0$, integrals around $C_\epsilon, C'_\epsilon, C_R \rightarrow 0$.

10.6/8



Need to define $f(z) = (1-z^2)^{-1/2}$ so that

$$f(-2) = + \frac{1}{i\sqrt{3}}$$

Now $(1-z^2)^{1/2} = \pm i(z^2-1)^{1/2} = \pm i(r_1 r_2)^{1/2} e^{i(\frac{\theta_1 + \theta_2}{2})}$

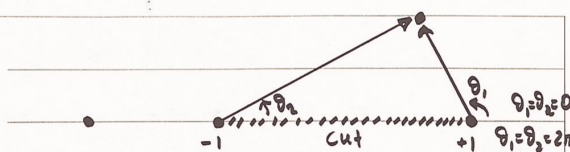
Choosing $0 \leq \theta_1, \theta_2 < 2\pi$ gives:

$$(1-z^2)^{1/2} \Big|_{-2} = \pm i\sqrt{3} e^{i2\pi/2} = \mp i\sqrt{3}$$

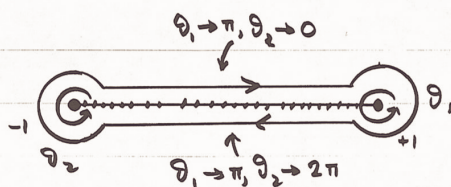
So, we will employ the definition

$$(1-z^2)^{1/2} = -i(z^2-1)^{1/2}, \quad 0 \leq \theta_1, \theta_2 < 2\pi$$

10.7/8



The cut extends only between $(-1, +1)$:
since both ϑ_1, ϑ_2 jump by 2π , $f(z)$ is
continuous on real axis, $x > +1$!



$$\begin{array}{l|l} \text{Top: } f(z) = -i(|z-1||z+1|)^{1/2} e^{i\frac{\pi+0}{2}} & \text{Bottom} \\ = \sqrt{1-x^2} \quad (+, \text{real}) & f(z) = -i(|z-1||z+1|)^{1/2} e^{i\frac{\pi+2\pi}{2}} \\ & = (-i)^2 \sqrt{1-x^2} = -\sqrt{1-x^2} \end{array}$$

10.8/8

$$\text{So: } \int_{C_2}^{\infty} \xrightarrow{\epsilon \rightarrow 0} \int_{-1}^{+1} \frac{dx}{(x+2)\sqrt{1-x^2}} = I$$

$$\int_{C_{II}}^{\infty} \xrightarrow{\epsilon \rightarrow 0} - \int_{+1}^{-1} \dots = I$$

Also

$$|\int_{C_R}^{\infty}| \leq \int_0^{2\pi} \frac{R d\vartheta}{(R-2)(R^2-1)^{1/2}} \xrightarrow{R \rightarrow \infty} 0$$

$$|\int_{C_\epsilon}^{\infty}| \leq \left| \int_{\pi}^{-\pi} \frac{\epsilon d\vartheta}{(2-\epsilon)\epsilon^{1/2} \cdot (1-\epsilon)^{1/2}} \right| \xrightarrow{\epsilon \rightarrow 0} 0; \text{ same for } C'_\epsilon$$

$$\text{So } 2I = \oint^{\infty} = 2\pi i \cdot \frac{1}{i\sqrt{3}} \Rightarrow I = \frac{\pi}{\sqrt{3}}$$