HMK #10

SEC. 5.1

#7a) $\sum_{k=0}^{\infty}\left(\frac{1+2i}{1-i}\right)^k$ using the ratio test $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(1+2i)^{k+1}}{(1-i)^{k+1}}\frac{(1-i)^k}{(1+2i)^k}\right| = \lim_{k\to\infty}\left|\frac{1+2i}{1-i}\right|$$

$$= \left|\frac{1+2i}{1-i}\right| = \sqrt{\frac{5}{2}} > 1 \text{ diverges.}$$

d) $\sum_{j=1}^{\infty}\frac{j!}{5^j}$ same thing use the ratio test.

$$\Rightarrow \lim_{j\to\infty}\left|\frac{(j+1)!}{5^{j+1}}\frac{5^j}{j!}\right| = \lim_{j\to\infty}\left|\frac{j+1}{5}\right| \to \infty \text{ diverges}$$

#10 $F_n(z) = \frac{z^n}{z^n-3^n}$

$|z|>3 \to F(z)\to 1$

$|z|<3 \to F(z)\to 0$

I will show 2 different ways of doing this problem

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for $|z|>3 \to F(z) \to 1$

1^{st} Method

$$F_n(z) = \frac{z^n}{z^n - 3^n} = z^n\left(\frac{1}{z^n - 3^n}\right) = \frac{z^n}{z^n}\left(\frac{1}{1 - \frac{3^n}{z^n}}\right) = \frac{1}{1 - \left(\frac{3}{z}\right)^n}$$

$$= 1 - \left(\frac{3}{z}\right)^n + \left(\left(\frac{3}{z}\right)^2\right)^n + \cdots$$

$|z|>3 \to 1$ as $n \to \infty$

2nd Method.

let $w = \left|\frac{3}{z}\right| < 1$ then

$$\left|F(z) - 1\right| = \left|\frac{1}{1 - \left(\frac{3}{z}\right)^n} - 1\right| = \left|\frac{\left(\frac{3}{z}\right)^n}{1 - \left(\frac{3}{z}\right)^n}\right|$$

$$\le \frac{w^n}{1 - w^n} < \varepsilon \Rightarrow w^n < \varepsilon(1 - w^n)$$

$$= w^n < \varepsilon - \varepsilon w^n$$

$$= w^n + \varepsilon w^n < \varepsilon$$

$$= w^n(1+\varepsilon) < \varepsilon$$

$\Rightarrow w^n < \frac{\varepsilon}{1+\varepsilon} \Rightarrow$ take the log on both sides.

$$n \log w < \log\left(\frac{\varepsilon}{1+\varepsilon}\right)$$

$$n > \frac{\log\left(\frac{\varepsilon}{1+\varepsilon}\right)}{\log w}$$

if $n > N = -\frac{\log\left(\frac{\varepsilon}{1+\varepsilon}\right)}{\log w} \Rightarrow |F(z) - 1| < \varepsilon \quad \forall \varepsilon > 0$

Now for $|z|<3 \to F(z) \to 0$

1st. $F_n = \dfrac{z^n}{z^n-3^n} = z^n\left(\dfrac{1}{z^n-3^n}\right) = -\dfrac{z^n}{3^n}\left(\dfrac{1}{1-\frac{z^n}{3^n}}\right) = -\dfrac{z^n}{3^n}\left(1+\left(\dfrac{z}{3}\right)^n + \dots\right.$

$|z|<3 \to 0$ as $n \to \infty$

2nd. $|z|<3 \quad F(z) \to 0$

let $w = \left|\dfrac{z}{3}\right| < 1$

$$|F(z)| = \frac{\left(\frac{z}{3}\right)^n}{1-\left(\frac{z}{3}\right)^n} \leq \frac{w^n}{1-w^n} < \varepsilon$$

$$\Rightarrow n > \frac{\log\left(\frac{\varepsilon}{1+\varepsilon}\right)}{\log w}.$$

#11b) $\displaystyle\sum_{k=0}^{\infty} \frac{(z-i)^k}{2^k}$ using the ratio test $\displaystyle\lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right|$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(z-i)^{k+1}}{2^{k+1}}\frac{z^k}{(z-i)^k}\right| = \lim_{k\to\infty}\left|\frac{(z-i)}{2}\right|$$

$$= \left|\frac{z-i}{2}\right| < 1 \quad \Rightarrow |z-i| < 2$$

#20. for uniform convergence we must show that for any $\varepsilon > 0$ we can find N such that if $n > N$, then

$$|S_n(z) - S(z)| < \varepsilon \quad \forall z \in T.$$

#20 cont.

Let $|z| = r < 1$

$$|S_n(z) - S(z)| = \left| \frac{1-z^{n+1}}{1-z} - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|}$$

$$\leq \frac{|z|^{n+1}}{1-|z|} \leq \frac{r^{n+1}}{1-r}$$

For this to be less than some given $\varepsilon > 0$ we have $\frac{|z|^{n+1}}{1-|z|} < \varepsilon \Rightarrow |z|^{n+1} < \varepsilon(1-|z|)$

$$\Rightarrow (n+1)\log|z| < \log(\varepsilon(1-|z|))$$

But $\log|z| < 0$

$$\Rightarrow n > \frac{\log(\varepsilon(1-|z|))}{\log|z|} - 1 = N$$

Now, if we only consider $|z| < r$ then we have that if $N = \frac{\log(\varepsilon(1-r))}{\log r} - 1$

then, since $\frac{\log(\varepsilon(1-r))}{\log r} > \frac{\log(\varepsilon(1-|z|))}{\log|z|}$

that if $n > N$, $|S_n(z) - S_n(z)| < \varepsilon \ \forall z$ as desired

But if we consider any $|z| < 1$, then the values of N depends on $|z|$ ($\to \infty$ as $|z| \to 1$)

$\therefore$ So convergence is not uniform

SEC. 5.2

#2. a) $z \in \mathbb{C}$

b) $z \in \mathbb{C}$

c) $z \in \mathbb{C}$

d) if you do the ratio test you should get

$$\left|\frac{z-i}{1-i}\right| < 1$$

$$\therefore |z-i| < \sqrt{2}$$

e) $\log(1-z)$

$$\therefore |z| < 1$$

f) $z \in \mathbb{C}$

#5e) $\frac{1+z}{1-z}$ around $z_0 = i$

$f(z) = \frac{1+z}{1-z}$ $\qquad$ $f(z_0) = \frac{1+i}{1-i} = i$

$f'(z) = \frac{1}{1-z} + \frac{1+z}{(1-z)^2} = \frac{2}{(1-z)^2}$ $\qquad$ $f'(z_0) = \frac{2}{(1-i)^2}$

$f''(z) = \frac{4}{(1-z)^3}$ $\qquad$ $f''(z_0) = \frac{4}{(1-i)^3}$

$f'''(z) = \frac{12}{(1-z)^3}$ $\qquad$ $f'''(z_0) = \frac{12}{(1-i)^4}$

$\vdots$ $\qquad$ $\vdots$

So if $f(z) = \left(\frac{1+z}{1-z}\right)$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$= i + \frac{2(z-i)}{(1-i)^2} + \frac{4(z-i)^2}{2!(1-i)^2} + \cdots$$

$$= i + \sum_{n=1}^{\infty} \frac{2(z-i)^n}{(1-i)^{n+1}} \qquad |z-i| < \sqrt{2} \checkmark$$

#5c) Another way

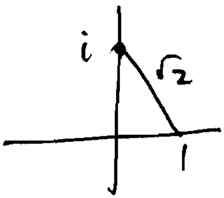
$$\frac{(1+i)+(z-i)}{(1-i)-(z-i)} = \frac{(1+i)+(1-i)-\left[(1-i)-(z-i)\right]}{(1-i)-(z-i)}$$

$$= -1 + \frac{z}{(1-i)-(z-i)}$$

$$= -1 + \frac{z}{1-i}\cdot\frac{1}{1-\frac{z-i}{1-i}}$$

$$\left|\frac{z-i}{1-i}\right| < 1$$

$$= 1 + \frac{z}{1-i}\left\{1+\left(\frac{z-i}{1-i}\right)+\left(\frac{z-i}{1-i}\right)^2+\cdots\right\}$$



$$|z-i| < \sqrt{2}$$

#11b) $\frac{e^z}{z-1}$

using Thm 6 on page 248.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1 + z + z^2 + z^3 + \cdots)$$

$$\Rightarrow e^z\left(-\frac{1}{1-z}\right) = \left(1 + z + \frac{z^2}{2!} + \cdots\right)\left(-1 - z - z^2 - \cdots\right)$$

$$= -1 - z - z^2 - z^3 - z - z^2 - z^3 - z^4 - \frac{z^2}{2} - \frac{z^3}{2} - \frac{z^3}{6}$$

$$= -1 - 2z - \frac{5}{2}z^2 - \frac{8}{3}z^3 - \cdots$$

SEC. 5.3

#3d) $\sum_{k=0}^{\infty} \frac{(-1)^k k}{3^k}(z-i)^k$

using the ratio test : $\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+1)(z-i)^{k+1}\,3^k}{3^{k+1}(-1)^k\,k\,(z-i)^k}\right| = \lim_{k\to\infty}\left|\frac{(-1)(k+1)(z-i)}{3\,k}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|\frac{k+1}{k}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|\frac{\cancel{k}(1+\frac{1}{k})}{\cancel{k}}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|1+\underbrace{\cancel{\frac{1}{k}}}_{0}\right|$$

Radius of convergence

$\therefore |z-i| < 3$

#5c $\oint_{\Gamma} e^{z} f(z)\, dz$

$\Gamma \Rightarrow |z|=1$

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{k^3}{3^k}\right) z^k$$

using the ratio test to see where the convergence is.

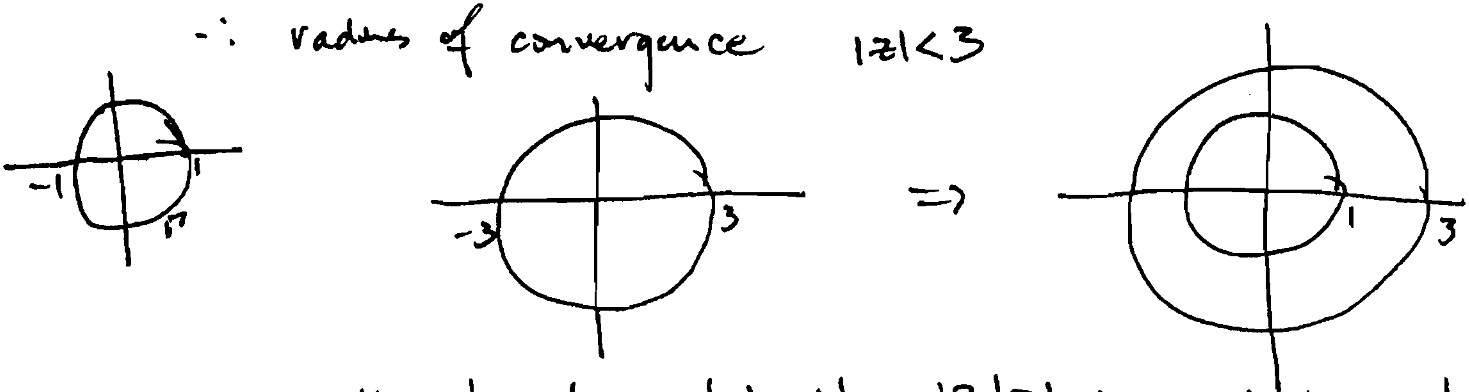
$$\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| \Rightarrow \lim_{k\to\infty} \left| \frac{(k+1)^3 z^{k+1}}{3^{k+1}} \frac{3^k}{k^3 z^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{z}{3} \frac{(k^3+3k^2+3k+1)}{k^3} \right|$$

$$= \frac{z}{3} \lim_{k\to\infty} \left| \cancel{\frac{k^3}{k^3}} \left(1 + \underbrace{\frac{3}{k}}_{0} + \underbrace{\frac{3}{k^2}}_{0} + \underbrace{\frac{1}{k^3}}_{0}\right) \right|$$

$$= \frac{z}{3}$$

$\therefore$ radius of convergence $|z|<3$



since around the boundary & inside $|z|=1$ is analytic also e^z is entire

$\oint_{|z|=1} e^{z} f(z)\, dz = 0$ by Cauchy's integral theorem.