#3. $\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$

$z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$= iz\,d\theta$

$\frac{dz}{iz} = d\theta$

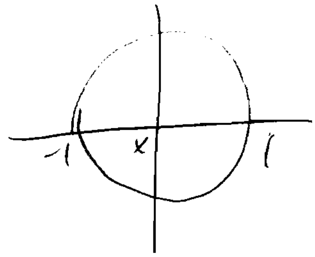
$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$= \frac{z+z^{-1}}{2}$

$= \frac{z^2+1}{2z}$

consider $\frac{1}{2}\int_0^{2\pi}\frac{dz}{iz}\left(\frac{1}{\left(3+2\left(\frac{z^2+1}{2z}\right)\right)^2}\right)$

$= \frac{1}{2i}\int_0^{2\pi}\frac{dz}{z}\,\frac{1}{\left(3+\frac{z^2+1}{z}\right)^2}$



$= \frac{1}{2i}\int_0^{2\pi}\frac{dz}{\cancel{z}}\,\frac{z^{\cancel{2}}}{(3z+z^2+1)^2}$

$= \frac{1}{2i}\int_0^{2\pi}\frac{dz\,z}{(z^2+3z+1)^2}$

$z^2+3z+1=0$

$z = \frac{-3\pm\sqrt{9-4}}{2}$

$= \frac{-3\pm\sqrt{5}}{2}$

using the method of residues

we have a pole of order 2

@ $z = \frac{-3\pm\sqrt{5}}{2}$

only $\frac{-3+\sqrt{5}}{2}$ is in the contour

Res @ $z = \frac{-3+\sqrt{5}}{2}$

$$\lim_{z\to\frac{-3+\sqrt{5}}{2}} \frac{d}{dz}\left[\left(z+\frac{3-\sqrt{5}}{2}\right)^2 \frac{z}{\left(z+\frac{3-\sqrt{5}}{2}\right)^2\left(z+\frac{3+\sqrt{5}}{2}\right)^2}\right]$$

$$= \lim_{z\to\frac{-3+\sqrt{5}}{2}} \frac{d}{dz}\left[\frac{z}{\left(z+\frac{3+\sqrt{5}}{2}\right)^2}\right] = \lim_{z\to-\frac{3+\sqrt{5}}{2}}\left(\frac{1}{\left(z+\frac{3+\sqrt{5}}{2}\right)^2}-\frac{2z}{\left(z+\frac{3+\sqrt{5}}{2}\right)^3}\right)$$

$$= \frac{1}{\left(\frac{-\cancel{3}+\sqrt{5}}{2}+\frac{\cancel{3}+\sqrt{5}}{2}\right)^2} - \frac{\left(\frac{-3+\sqrt{5}}{2}\right)2}{\left(\frac{-\cancel{3}+\sqrt{5}}{2}+\frac{\cancel{3}+\sqrt{5}}{2}\right)^3} = \frac{1}{5}+\frac{(3-\sqrt{5})}{5\sqrt{5}}$$

$$= \frac{\sqrt{5}+3-\sqrt{5}}{5\sqrt{5}} = \frac{3}{5\sqrt{5}}$$

$$\frac{2}{i}\oint \frac{z\,dz}{(z^2+3z+1)^2} = 2\pi i \cdot \frac{1}{2i}\sum \text{Res}$$

$$= \pi\left(\frac{3}{5\sqrt{5}}\right)$$

$$= \frac{\pi\cdot 3}{5\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{3\pi\sqrt{5}}{25}$$

#6. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta}\,d\theta = \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$ $\quad a > |b| > 0$

$\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ $\qquad \cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

let $z = e^{i\theta}$

$\dfrac{dz}{iz} = d\theta$

$\sin^2\theta = \left(\dfrac{z-z^{-1}}{2i}\right)^2$ $\qquad \cos\theta = \dfrac{z+\frac{1}{z}}{2} = \dfrac{z^2+1}{2z}$

$= \left(\dfrac{z^2-1}{2iz}\right)^2$

Consider

$$\Rightarrow \oint \frac{(z^2-1)^2}{-4z^2}\,\frac{1}{\left(a+b\left(\frac{z^2+1}{2z}\right)\right)}\,\frac{dz}{iz}$$

$$= -\frac{1}{4i}\oint \frac{(z^2-1)^2}{z^{\cancel{3}2}}\,\frac{\cancel{2}z}{2az+bz^2+b}$$

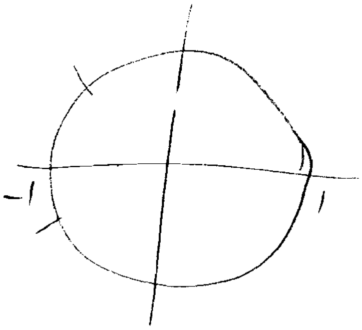
$$= -\frac{1}{2i}\oint \frac{(z^2-1)^2}{z^2}\,\frac{1}{bz^2+2az+b}\,dz$$

double pole @ $z=0$

pole @ $bz^2+2az+b=0$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$



the only one we want is $\frac{-a+\sqrt{a^2-b^2}}{b}$ for $a > |b| > 0$ which was given of z^2

Res @ $z=0$

$$\lim_{z\to 0} \frac{d}{dz}\left(z^2 \frac{(z^2-1)}{\cancel{z^2}(bz^2+2az+b)}\right)$$

$$= \lim_{z\to 0} \frac{2z}{(bz^2+2az+b)} - \frac{(\cancel{z^2}-1)(2\cancel{bz}+2a)}{(b\cancel{z^2}+2a\cancel{z}+b)^2}$$

$$= -\frac{2a}{b^2}$$

Res @ $z = \frac{-a+\sqrt{a^2+b^2}}{b}$

$$\lim_{z\to\frac{-a+\sqrt{a^2-b^2}}{b}} \left[\left(z+\frac{a-\sqrt{a^2+b^2}}{b}\right) \frac{(z^2-1)}{\left(z+\frac{a-\sqrt{a^2+b^2}}{b}\right)\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)}\right]$$

$$\lim_{z\to\frac{-a+\sqrt{a^2-b^2}}{b}} \left(\frac{z^2-1}{z+\frac{a+\sqrt{a^2-b^2}}{b}}\right)$$

$$\frac{\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)^2-1}{\frac{-a+\sqrt{a^2-b^2}}{b}+\frac{a+\sqrt{a^2-b^2}}{b}} = \frac{\frac{a^2+a^2-b^2-2a\sqrt{a^2-b^2}}{b^2}-\frac{b^2}{b^2}}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$\Rightarrow \frac{\cancel{2}a^2 - \cancel{2}b^2 - \cancel{2}a\sqrt{a^2-b^2}}{\cancel{b^2}} \cdot \frac{\cancel{b}}{2\sqrt{a^2-b^2}}$$

$$= \frac{a^2-b^2-a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}}$$

$$\therefore \oint -\frac{1}{2i} \frac{(z^2-1)^2}{z^2} \frac{1}{bz^2+2az+b}\,dz = 2\pi i\left(-\frac{1}{\cancel{2i}}\right)\sum \text{Res}$$

$$\Rightarrow = -\pi\left(-\frac{2a}{b^2} + \frac{a^2-b^2-a\sqrt{a^2-b^2}}{b\sqrt{a^2-b^2}}\right)$$

$$= -\pi\left(\frac{-2a\sqrt{a^2-b^2} + a^2b - b^3 - ab\sqrt{a^2-b^2}}{b^2\sqrt{a^2-b^2}}\right)$$

$$= -\pi\left(\frac{-a\cancel{\sqrt{a^2-b^2}}(2+b) + b(a^2-b^2)}{b^2\cancel{\sqrt{a^2-b^2}}}\right)$$

$$= -\pi\left(\frac{-2a-ab}{b^2} + \frac{b\sqrt{a^2-b^2}}{b^2}\right)$$

$$= -\pi\left(\frac{-2a - ab + b\sqrt{a^2-b^2}}{b^2}\right)$$

$$= \frac{\pi}{b^2}\left(2a + ab - b\sqrt{a^2-b^2}\right)$$

$$= \frac{\pi}{b^2}\left(2a + b\left(a - \sqrt{a^2-b^2}\right)\right)$$

11. $\int_0^{\pi} \tan(\theta + i\theta)\, d\theta = \pi i \operatorname{sign} a$ a real # nonzero

$$\int_0^{\pi} \frac{\sin(\theta + i\theta)}{\cos(\theta + i\theta)}\, d\theta$$

$$= -- \int_0^{\pi} \frac{\sin(\theta + i\theta)}{\cos(\theta + i\theta)}\, d\theta$$

$\theta + i\theta = z$

$\theta(1+i)$

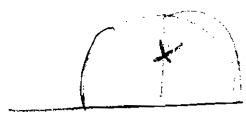
$d\theta = \frac{dz}{(1+i)}$

$$\frac{1}{(1+i)} \int_0^{2\pi} \frac{\sin z}{\cos z}\, dz$$

consider

$$= - \frac{1}{(1+i)} \oint \frac{z^2-1}{\cancel{2}i\cancel{z}} \cdot \frac{\cancel{2z}}{z^2+1}\, dz$$

$$= \frac{1}{(1+i)} \frac{1}{i} \oint \frac{z^2-1}{z^2+1}\, dz$$



$z^2+1=0$

$z = \pm i$

Res @ $z = i$

$$\lim_{z \to i} (z-i) \frac{z^2-1}{(z+i)(z-i)} = \frac{-2}{2i} = -\frac{1}{i}$$

$$\oint \frac{z^2-1}{z^2+1}\, dz = 2\pi i \left(\frac{1}{1+i}\right)\left(\frac{1}{i}\right)\left(-\frac{1}{i}\right)$$

$$= 2\pi i \left(\frac{1}{1+i}\right)\left(\frac{1-i}{1-i}\right)$$

$$= \frac{2\pi i\,(1-i)}{2} = \pi i(1-i)$$

$$= \pi i + \pi$$

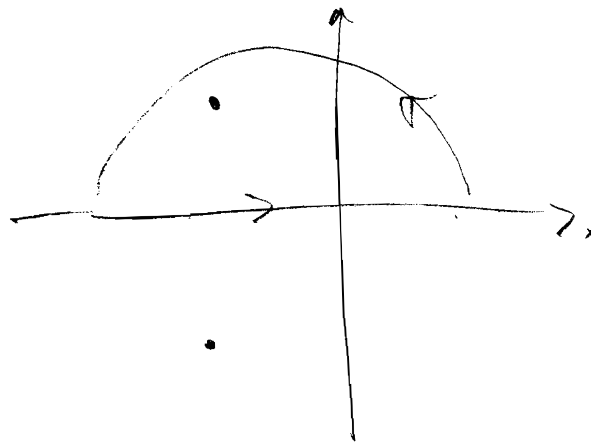
1. Ex. $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi$ $Q > P$

$z = \frac{-2 \pm \sqrt{4-4(2)}}{2}$

$= -1 \pm i$

Consider

$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2}$



$f(z) = \frac{1}{z^2+2z+2}$

$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2} = 2\pi i \sum \text{Res}.$

Res @ $z = -1+i$

$\lim_{z \to -1+i} \cancel{(z+1-i)} \frac{1}{\cancel{(z+1-i)}(z+1+i)}$

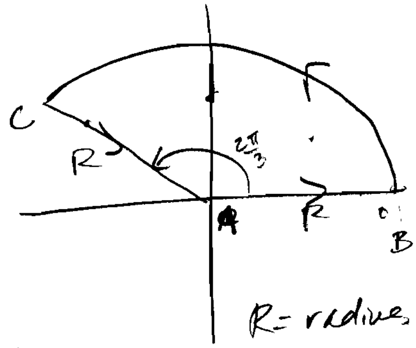
$\lim_{z \to -1+i} \frac{1}{[(-1+i)+1+i]} = \frac{1}{2i}$

$\text{Res}_1 = \frac{1}{2i}$

$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2} = 2\pi i \ \text{Res} @ z = -1+i$

$= 2\pi i \left(\frac{1}{2i}\right) = \pi$ ✓

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$



Consider: $\oint \frac{dz}{z^3+1}$

singularities occur @ $z^3=-1$

$$= e^{i\frac{\pi}{2}}$$

$$z = e^{i\frac{\pi}{6}},\ e^{i\frac{3\pi}{6}},\ e^{i\frac{5\pi}{6}},\ e^{i\frac{7\pi}{6}}$$

poles lie in $e^{i\frac{\pi}{6}}$ & $e^{i\frac{3\pi}{6}}$

we know that by Cauchy's Thm

$$\oint \frac{dz}{z^3+1} = 0$$

$$\int_{AB} + \int_{BC} + \int_{CA} = 0$$

rom AB $z = x$ from $x=0$ to $x=R$ $\qquad dz = dx$

BC $z = Re^{i\theta}$ from $\theta=0$ to $\theta=$ $\qquad dz = Rie^{i\theta}d\theta$

CA $z = e^{2\pi i/3}r$ from $r=R$ to $r=0$ $\qquad dz = e^{2\pi i/3}dr$

$$\int_{AB} \frac{dx}{x^3+1} + \int_{BC} \frac{Rie^{i\theta}d\theta}{(Re^{i\theta})^3+1} + \int_{CA} \frac{rie^{i\theta}d\theta}{(re^{i\theta})^3+1} = 0$$

Cauchy's Residue Thm

$$= -\frac{2\pi i}{3} e^{\pi i/3}$$

ntgral using Cauchy inequality

$$|1+z^3| \geq |z|^n - 1 = R^n - 1$$

$$\left| \int_{BC} \frac{dz}{1+z^3} \right| \leq \int_{BC} \frac{|dz|}{|1+z^3|} \leq \frac{1}{R^3-1} \int |dz| \to 0 \quad R \to \infty$$

A $\displaystyle\int_{C_A} \frac{dz}{1+z^3} = \int_{R}^{0} \frac{e^{2\pi i/3}}{1+(re^{2\pi i/3})^3}\,dr$

$$= -\int_0^R \frac{e^{2\pi i/3}}{1+r^3}\,dr$$

$$= -\frac{2\pi i}{3} e^{\pi i/3}$$

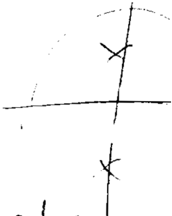
Since $1-e^{2\pi i/3} \neq 0$

$$\int_0^\infty \frac{dx}{1+x^3} = -\frac{2\pi i}{3} \frac{e^{\pi i/3}}{1-e^{2\pi i/3}}$$

$$= \frac{2\pi i/3}{e^{\pi i/3}-e^{-i\pi/3}} = \frac{2\pi i}{3}\frac{1}{\sqrt{3}i}$$

$$= \frac{2\pi\sqrt{3}}{9} \checkmark$$

13. show $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{\pi\,(2n)!}{2^{2n}(n!)^2}$



singularity $x^2 = -1$

$x = \pm i$ where it will have $n+1$ poles!

we can just use the $+i$ for it is in the contour

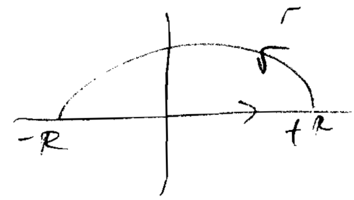
Consider

$$\oint \frac{dz}{(1+z^2)^{n+1}} \qquad n = 0, 1, 2, \ldots$$

$$2 \quad \text{p.v.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$$

consider

$$\oint_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz$$



singularities occur @ $z^2-2z+10=0$

$$z = \frac{2 \pm \sqrt{4-4(10)}}{2} = \frac{2 \pm 6i}{2} =$$

since I have drawn the contour I will only use $z = 1+3i$

Res. @ $z = 1+3i$

$$\lim_{z \to 1+3i} \left[\cancel{(z-1-3i)} \frac{z e^{iz}}{\cancel{(z-1-3i)}(z-1+3i)} \right] = \lim_{z \to 1+3i} \frac{z e^{iz}}{z-1+3i}$$

$$= \frac{(1+3i)\left(e^{i(1+3i)}\right)}{(\cancel{1}+3i)-\cancel{1}+3i)} = \frac{1+3i}{6i}\left(e^{i} e^{-3}\right)$$

$$\oint \frac{z e^{iz}}{z^2-2z+10}\,dz = 2\pi i \left(\frac{1+3i}{6i}\right)\left(e^{i} \cdot e^{-3}\right)$$

$$= \frac{\pi}{3}(1+3i)\left(e^{i} \cdot e^{-3}\right)$$

Now break up the integral

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2-2x+10}\,dx + \int_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz = \frac{\pi}{3e^3}(1+3i)e^{i}$$

$$\int_{-R}^{R} \frac{x \cos x}{x^2-2x+10}\,dx + i\int_{-R}^{R} \frac{x \sin x}{x^2-2x+10}\,dx + \int_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz = \frac{\pi}{3e^3}(1+$$

aking $\lim_{R\to\infty}$ the 3^{rd} integral $\to 0$

$$\Rightarrow \int_{-R}^{R} \frac{x\cos x}{x^2-2x+10}\,dx + i\int_{-R}^{R} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(1+3i)e^{i}$$

$$= \frac{\pi}{3e^3}(1+3i)(\cos 1 + i\sin$$

$$= \frac{\pi}{3e^3}\cos 1 - 3\sin 1$$

$$+ \frac{\pi}{3e^3}(3i\cos 1 + i$$

$$\int_{-R}^{R} \frac{x\cos x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\cos 1 - 3\sin 1$$

$$\int_{-R}^{R} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$$

#8. $\int_0^\infty \frac{x^3 \sin 2x}{(x^2+1)^2}\, dx$

consider

$$\oint \frac{z^3 e^{2iz}}{(z^2+1)^2}\, dz$$

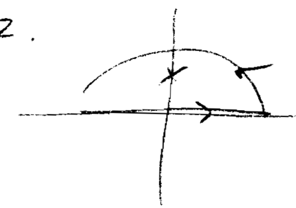
$z^2+1=0$

$z^2=-1$

$z=\pm i$ ← singularities occur

pole order 2.

use only $+i$ which is in the contour



Res. @ $z=i$

$$\lim_{z\to i} \frac{d}{dz}\left[\frac{(z-i)^2\, z^3 e^{2iz}}{(z-i)^2(z+i)^2}\right]$$

$$\lim_{z\to i} \frac{d}{dz}\left[\frac{z^3 e^{2iz}}{(z+i)^2}\right] = \lim_{z\to i} \frac{3z^2 e^{2iz}}{(z+i)^2} + \frac{2iz^3 e^{2iz}}{(z+i)^2} - \frac{2z^3 e^{2i}}{(z+i)}$$

$$\frac{+3e^{-2}}{+4} + \frac{2i(+i)e^{-2}}{+4} + \frac{2(-i)e^{-2}}{8i}$$

$$= \frac{3e^{2}}{4} - \frac{2e^{-2}}{4} - \frac{e^{-2}}{4} = 0 \ ?$$