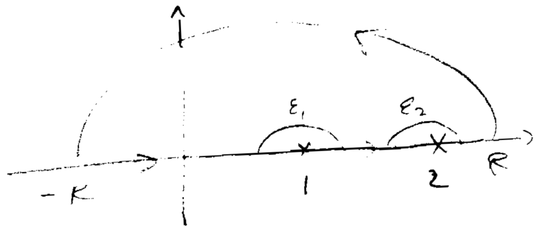
344

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 3x + 2}\, dx = \pi[\sin 1 - 2 \sin 2]$$

Consider

$$\oint \frac{z e^{iz}}{z^2 - 3z + 2}\, dz$$



finding poles $\quad z^2 - 3z + 2 = 0$

$$(z-2)(z-1) = 0$$

$$z = 2, 1$$

$$\lim_{\substack{R \to \infty \\ \varepsilon_1, \varepsilon_2 \to 0}} \left(\int_{-R}^{1-\varepsilon_1} + \int_{1+\varepsilon_1}^{2-\varepsilon_2} + \int_{2+\varepsilon_2}^{R} \right) \frac{z e^{iz}}{z^2 - 3z + 2}\, dz$$

$$f(z) = \frac{z e^{iz}}{z^2 - 3z + 2}$$

using Jordan's lemma $\quad \lim_{R \to \infty} J_R = 0$

$$\lim_{\varepsilon_1 \to 0} J_{\varepsilon_1} = -\pi i \operatorname{Res}(1) = -\pi i \, (z-1) f(z) =$$

$$= -\pi i \lim_{z \to 1} \frac{z e^{iz}}{z - 2} = \frac{+i\pi e^{i}}{+1}$$

$$\lim_{\varepsilon_2 \to 0} J_{\varepsilon_2} = -\pi i \operatorname{Res}(2) = -\pi i \, (z-2) f(z) =$$

$$= -\pi i \lim_{z \to 2} \frac{z e^{iz}}{z - 1} = \frac{2 e^{2i}}{+1} i\pi$$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 - 3x + 2}\, dx = i\pi e^{i} + 2 e^{2i} i\pi$$

$$= i\pi (e^{i} + 2 e^{2i})$$

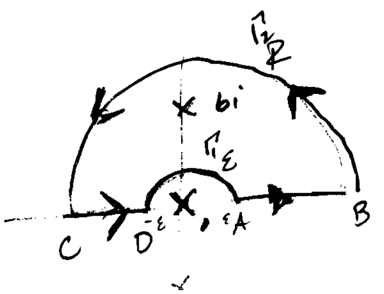
$$= i\pi (\cos 1 + i \sin 1 + 2 \cos 2 + 2i \sin 2)$$

$$= i\pi \cos 1 - \pi \sin 1 + 2 i\pi \cos 2 - 2\pi \sin 2$$

$$\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 3x + 2}\, dx = -\pi (\sin 1 + 2 \sin 2)$$

#12 a>0 b>

$$\int_0^\infty \frac{\sin ax\, dx}{x(x^2+b^2)} = \frac{\pi}{2b^2}\left(1-e^{-ab}\right)$$



pole simple @ $x=0$

$x^2+b^2=0$

$x^2=-b^2$

$x=\pm bi$ ← pole.

Consider

$$\oint \frac{e^{iza}dz}{z(z^2+b^2)}$$

Res @ $z=bi$

$$\lim_{z\to bi} (z-bi)\frac{e^{iza}}{z(z-bi)(z+bi)} = \frac{e^{iza}}{z(z+bi)} = \frac{e^{ibia}}{bi(2bi)} = \frac{e^{-ba}}{-2b^2}$$

By residue Thm:

$$\oint \frac{e^{iza}dz}{z(z^2+b^2)} = 2\pi i \sum \text{Res} = -\frac{\pi i e^{-ba}}{b^2}$$

By cauchys thm:

$$\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA} = -\frac{\pi i e^{-ba}}{b^2}$$

$$\int_\varepsilon^R \frac{e^{iax}}{x(x^2+b^2)}dx + \int_{BC} \frac{e^{iaz}}{z(z^2+b^2)}dz + \int_{-R}^{-\varepsilon} \frac{e^{iax}}{x(x^2+b^2)}dx + \int_{DA} \frac{e^{iaz}}{z(z^2+b^2)}dz = -\frac{\pi i e^{-ab}}{b^2}$$

replace $-x$ by x on 3rd integral

$$\int_\varepsilon^R \frac{e^{iax}-e^{-iax}}{x(x^2+b^2)}dx + \int_{BC} + \int_{DA} = -\frac{\pi i e^{-ab}}{b^2}$$

taking $R\to\infty$ the $\int_{BC} \to 0$

#12 cont.

$$\Rightarrow \int_{\varepsilon}^{R} \frac{e^{iax} - e^{-iax}}{x(x^2+b^2)}\,dx + \int_{DA} = -\frac{\pi i e^{-ba}}{b^2}$$

$$z = \varepsilon e^{i\theta}$$

$$dz = \varepsilon i e^{i\theta}\,d\theta$$

$$\int_{BC} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = \lim_{\varepsilon\to 0} \int \frac{e^{ia\varepsilon e^{i\theta}}\,(\cancel{\varepsilon i e^{i\theta}})\,d\theta}{(\cancel{\varepsilon e^{i\theta}})(\varepsilon e^{2i\theta} + b^2)}$$

$$\Rightarrow \int_{\pi}^{0} \frac{i\,d\theta}{b^2} = -\frac{\pi i}{b^2}$$

$$\therefore\ 2i \int \frac{\sin ax}{x(x^2+b^2)}\,dx = +\frac{\pi i}{b^2} - \frac{\pi i e^{-ab}}{b^2}$$

$$= \frac{1}{2i}\left(\frac{\pi i}{b^2}\left(1 - e^{-ab}\right)\right)$$

$$= \frac{\pi}{2b^2}\left(1 - e^{-ab}\right)$$

$$\therefore \int_0^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx = \frac{\pi}{2b^2}\left(1 - e^{-ab}\right) ✓$$