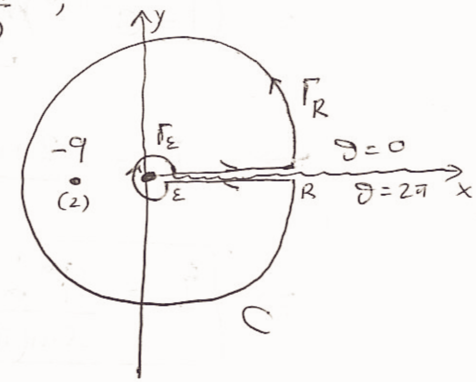
6.6.3 $\quad I = \int_0^\infty \frac{x^\alpha\,dx}{(x+9)^2} = \frac{9^{\alpha-1}\,\pi\alpha}{\sin(\pi\alpha)}, \quad 0<|\alpha|<1$



$$\oint_C \frac{z^\alpha\,dz}{(z+9)^2} = 2\pi i \,\mathrm{Res}\,(z=-9)$$

Double pole at -9; define branch for z^α as shown. Then

$$\mathrm{Res}\,(z=-9) = \lim_{z\to -9} \frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}\Big|_{z=-9=9e^{i\pi}}$$

$$= \alpha\,(9e^{i\pi})^{\alpha-1} = 9^{\alpha-1}\alpha\, e^{i\pi(\alpha-1)}$$

As $\varepsilon\to 0$, $R\to\infty$

$$\left|\int_{\Gamma_\varepsilon} \frac{z^\alpha\,dz}{(z+9)^2}\right| \le \int_0^{2\pi} \frac{|\varepsilon^\alpha e^{\alpha i\vartheta}\;\varepsilon i e^{i\vartheta}|\,d\vartheta}{|(\varepsilon e^{i\vartheta}+9)^2|} \le$$

$z=\varepsilon e^{i\vartheta}$

$$\le \frac{\varepsilon^{\alpha+1}}{(a-\varepsilon)^2}\,2\pi \xrightarrow{\varepsilon\to 0} 0 \quad \text{if } \alpha>-1$$

$$\left|\int_{\Gamma_R} \frac{z^\alpha\,dz}{(z+9)^2}\right| \le \int_0^{2\pi} \frac{|R^\alpha e^{\alpha i\vartheta}\;R i e^{i\vartheta}|\,d\vartheta}{|(Re^{i\vartheta}+9)^2|}$$

$$\le \frac{R^{\alpha+1}}{(R-9)^2}\cdot 2\pi \xrightarrow{R\to\infty} 0 \quad \text{if } \alpha<1$$

$$\text{So} \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \oint \frac{z^\alpha dz}{(z+9)^2} = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \left(\int_\varepsilon^R \frac{x^\alpha\,dx}{(x+9)^2} + \int_R^\varepsilon \frac{x^\alpha e^{2\pi i\alpha}\,dx}{(x+9)^2}\right)$$

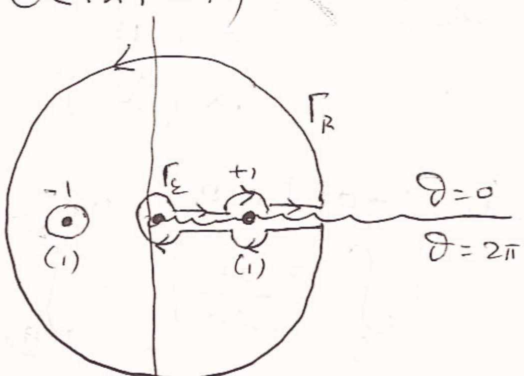
$$= \left(1-e^{2\pi i\alpha}\right) PV\int_0^\infty \frac{x^\alpha\,dx}{(x+q)^2} = 2\pi i \cdot q^{\alpha-1}\alpha\, e^{i\pi(\alpha-1)}$$

$$\Rightarrow I = \frac{2\pi i\cdot q^{\alpha-1}\alpha\, e^{i\pi\alpha}e^{-i\pi}}{e^{i\pi\alpha}\left(e^{-i\pi\alpha}-e^{i\pi\alpha}\right)} =$$

$$\Rightarrow \boxed{I = \frac{q^{\alpha-1}\alpha\pi}{\sin(\pi\alpha)}}$$

$\triangleright$

6.6.6 $I = PV\int_0^\infty \frac{x^\alpha}{x^2-1}dx \qquad (0<|\alpha|<1)$



By introducing indented ~~[illegible]~~ keyhole contour:

$$\oint_C \frac{z^\alpha}{z^2-1}dz =$$

$$= 2\pi i\left(\mathrm{Res}(-1) + \tfrac{1}{2}R(1,\text{top}) + \tfrac{1}{2}\mathrm{Res}(1,\text{bottom})\right)$$

With branch for z^α as shown:

$$\mathrm{Res}(-1) = \mathrm{Res}(e^{i\pi}) = \lim_{z\to e^{i\pi}}\left(\frac{z^\alpha}{z-1}\right) = \frac{e^{i\alpha\pi}}{-2}$$

$$\mathrm{Res}(+1,\text{top}) = \mathrm{Res}(e^{i0}) = \lim_{z\to e^{i0}}\left(\frac{z^\alpha}{z+1}\right) = \frac{e^{i\alpha 0}}{1+1} = \frac{1}{2}$$

$$\mathrm{Res}(+1,\text{bottom}) = \mathrm{Res}(e^{i2\pi}) = \lim_{z\to e^{i2\pi}}\left(\frac{z^\alpha}{z+1}\right) = \frac{e^{i2\pi\alpha}}{2}$$

with $\left|\int_{\Gamma_\varepsilon} \frac{z^\alpha}{z^2-1}dz\right| \le \frac{\varepsilon^{\alpha+1}}{1-\varepsilon^2}\cdot 2\pi \xrightarrow{\varepsilon\to 0} 0 \quad (\alpha>-1)$

$$\left|\int_{\Gamma_R} \frac{z^\alpha}{z^2-1}dz\right| \le \frac{R^{\alpha+1}}{R^2-1}\cdot 2\pi \xrightarrow{R\to\infty} 0 \quad (\alpha<1)$$

we have

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \oint_C \frac{z^\alpha}{z^2-1}dz = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(PV\int_\varepsilon^R \frac{x^\alpha\,dx}{x^2-1} + PV\int_R^\varepsilon \frac{x^\alpha e^{2\pi i\alpha}\,dx}{x^2-1}\right)$$

$$(1-e^{2\pi i\alpha})\,PV\int_0^\infty \frac{x^\alpha\,dx}{x^2-1} = 2\pi i\sum \text{Res}$$

$$\Rightarrow I = \frac{2\pi i}{1-e^{2\pi i\alpha}}\cdot\left[\frac{1}{4}+\frac{1}{4}e^{2\pi\alpha i}-\frac{1}{2}e^{i\alpha\pi}\right]$$

$$= \frac{\pi}{e^{i\pi\alpha}\left(\frac{e^{-i\pi\alpha}-e^{i\pi\alpha}}{2i}\right)}\frac{1}{4}\cdot(1-e^{i\alpha\pi})^2$$

$$= -\frac{\pi}{\sin\pi\alpha}\cdot\left(i\,\frac{e^{-i\pi\alpha/2}-e^{i\alpha\pi/2}}{2i}\right)^2 = +\frac{\pi}{\sin\pi\alpha}\left(\sin^2\frac{\pi\alpha}{2}\right)$$

$$\sin^2\frac{\pi\alpha}{2} = \frac{1-\cos\pi\alpha}{2}$$

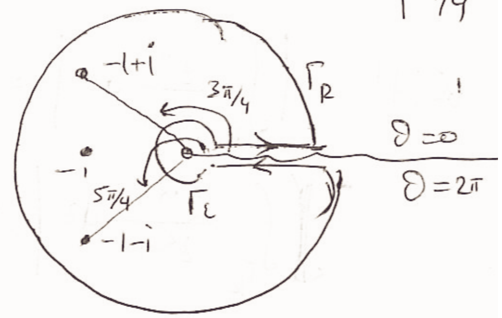
$$= \frac{\pi(1-\cos\pi\alpha)}{2\sin\pi\alpha}$$

(6.6.9a) $I=\int_0^\infty \frac{x\,dx}{(x+1)(x^2+2x+2)}$

$$\oint_C \frac{z\log z\,dz}{(z+1)(z^2+2z+2)} = 2\pi i\left(\text{Res}(-1)+\text{Res}(-1+i)+\text{Res}(-1-i)\right)$$

$z^2+2z+2 = (z+1)^2+1$

$z=-1\pm i$



$$\operatorname{Res}(-1) = \frac{z \operatorname{Log} z}{z^2+2z+2}\Bigg|_{z=e^{i\pi}} = \frac{-1(i\pi)}{1-2+2} = -i\pi$$

$$\operatorname{Res}(-1+i) = \frac{z \operatorname{Log} z}{(z+1)(z+1+i)}\Bigg|_{z=\sqrt{2}e^{i3\pi/4}}$$

$$= \frac{(-1+i)\left[\frac{i}{2}\operatorname{Log} 2 + i\,\frac{3\pi}{4}\right]}{i\,(2i)} = \left(\frac{1-i}{2}\right)\left(\frac{1}{2}\operatorname{Log} 2 + i\,\frac{3\pi}{4}\right)$$

$$\operatorname{Res}(-1-i) = \frac{z \operatorname{Log} z}{(z+1)(z+1-i)}\Bigg|_{\substack{z=\sqrt{2}e^{i5\pi/4} \\ =-1-i}} = \left(\frac{1+i}{2}\right)\left[\frac{1}{2}\operatorname{Log} 2 + i\,\frac{5\pi}{4}\right]$$

$$2\pi i \sum \operatorname{Res} = 2\pi i\left\{-\cancel{i\pi} + \frac{1}{2}\operatorname{Log} 2 + \cancel{i\pi} - \frac{\pi}{4}\right\} = \pi i\left(\operatorname{Log} 2 - \frac{\pi}{2}\right)$$

Now $\displaystyle\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \oint = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^R \frac{x \operatorname{Log} x}{(x+1)(x^2+2x+2)}\,dx + \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_R^\varepsilon \frac{x(\operatorname{Log} x + 2\pi i)}{(x+1)(x^2+2x+2)}\,dx$

$$= -2\pi i \int_0^\infty \frac{x\,dx}{(x+1)(x^2+2x+2)} = -2\pi i\, I$$

so $\boxed{I = \frac{1}{2}\left(\frac{\pi}{2} - \operatorname{Log} 2\right)}$

since $\displaystyle\left|\int_{\Gamma_R}\right| \le M\,\frac{R^2(\operatorname{Log} R - 2\pi)}{(R-1)(R^2-2R-2)} \xrightarrow{R\to\infty} 0$

$$\left|\int_{\Gamma_\varepsilon}\right| \le m\,\frac{\varepsilon^2(|\operatorname{Log}\varepsilon| - 2\pi)}{(1-\varepsilon)(2-2\varepsilon-\varepsilon^2)} \xrightarrow{\varepsilon\to 0} 0$$