HMK #10

SEC. 5.1

7a) $\sum_{k=0}^{\infty} \left(\frac{1+2i}{1-i} \right)^k$

using the ratio test: $\lim_{k \to \infty} \left| \frac{C_{k+1}}{C_k} \right|$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(1+2i)^{k+1}}{(1-i)^{k+1}} \frac{(1-i)^k}{(1+2i)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{1+2i}{1-i} \right| = \left| \frac{1+2i}{1-i} \right| = \sqrt{\frac{5}{2}} \leq 2 \quad \text{diverges.}$$

d) $\sum_{j=1}^{\infty} \frac{j!}{5^j}$

using the ratio test

$$\Rightarrow \lim_{j\to\infty} \left| \frac{(j+1)!}{5^{j+1}} \frac{5^j}{j!} \right|$$

$$= \lim_{j\to\infty} \left| \frac{j+1}{5} \right| \to \infty \quad \text{diverges.}$$

#10. $F_n(z) = \frac{z^n}{z^n - 3^n}$

$$= z^n \left(\frac{1}{z^n - 3^n} \right)$$

$$= \frac{z^n}{z^n} \left(\frac{1}{1 - \frac{3^n}{z^n}} \right)$$

$$= \frac{1}{1 - \frac{3^n}{z^n}}$$

$$= 1 - \frac{3}{z} + \frac{3^2}{z^2} + \cdots \quad |z| > 3$$

Note: $\frac{1}{1-z} = 1 + z + z^2 + \cdots \quad |z| <$

$\frac{1}{1+z} = 1 - z - z^2 + \cdots \quad |z| <$

$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \quad |z| > 1$

$\frac{1}{1+\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{z^2} + \cdots \quad |z| > 1$

#10 CONT.

$$F_n(z) = \frac{z^n}{z^n - 3^n}$$

$$= -\frac{z^n}{3^n - z^n} = -\frac{z^n}{3^n}\left(\frac{1}{1-\frac{z^n}{3^n}}\right)$$

$$= -\frac{z^n}{3^n}\left(\frac{1}{1-\left(\frac{z}{3}\right)^n}\right)$$

$$= -\frac{z^n}{3^n}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \cdots\right) \qquad |z| < 1$$

#11b) $\displaystyle\sum_{k=0}^{\infty} \frac{(z-i)^k}{2^k}$

Ratio test: $\displaystyle\lim_{k\to\infty}\left|\frac{C_{k+1}}{C_k}\right|$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(z-i)^{k+1}\; z^k}{z^{k+1}\,(z-i)^k}\right| = \lim_{k\to\infty}\left|\frac{z-i}{2}\right| < 1$$

$$|z-i| < 2$$

#20.

SEC. 249

\# 2. a) $z \in \mathbb{C}$

b) $z \in \mathbb{C}$

c) $z \in \mathbb{C}$

d)

e)

\#5. e) $\frac{1+z}{1-z}$ around $z_0 = i$

$$f(z) = \frac{1+z}{1-z} \qquad f(z_0) = \frac{1+i}{1-i}$$

$$f'(z) = \frac{1}{1-z} + \frac{1+z}{(1-z)^2} \qquad f'(z_0) = \frac{2}{(1-i)^2}$$

$$= \frac{2}{(1-z)^2}$$

$$f''(z) = \frac{4}{(1-z)^3} \qquad f''(z_0) = \frac{4}{(1-i)^3}$$

$$f'''(z) = \frac{12}{(1-z)^4} \qquad f'''(z_0) = \frac{12}{(1-i)^4}$$

#11b) $\dfrac{e^z}{z-1}$

using Thm6 on page 248.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1 + z + z^2 + z^3 + \cdots)$$

$$\Rightarrow e^z\left(-\frac{1}{1-z}\right) = \left(1 + z + \frac{z^2}{2!} + \cdots\right)\left(-1 - z - z^2 - \cdots\right)$$

$$= -1 - z - z^2 - z^3 - z - z^2 - z^3 - z^4 - \frac{z^2}{2} - \frac{z^3}{2} - \frac{z^3}{6}$$

$$= -1 - 2z - \frac{5}{2}z^2 - \frac{8}{3}z^3 - \cdots$$

SEC. 5.3

#3d) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k k}{3^k}(z-i)^k$

using the ratio test: $\displaystyle\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$

$$\Rightarrow \lim_{k\to\infty}\left|\frac{(-1)^{k+1}(k+1)(z-i)^{k+1}\,3^k}{3^{k+1}(-1)^k\,k\,(z-i)^k}\right| = \lim_{k\to\infty}\left|\frac{(-1)\,(k+1)\,(z-i)}{3\,k}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|\frac{k+1}{k}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|\frac{\cancel{k}\left(1+\frac{1}{k}\right)}{\cancel{k}}\right|$$

$$= \frac{|z-i|}{3}\lim_{k\to\infty}\left|1 + \underbrace{\cancel{\tfrac{1}{k}}}_{0}\right|$$

Radius of convergence

$\therefore |z-i| < 3$

#5c $\oint_{\Gamma} e^{z} f(z)\, dz$

$\Gamma \Rightarrow |z|=1$

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{k^3}{3^k}\right) z^k$$

using the ratio test to see where the convergence is.

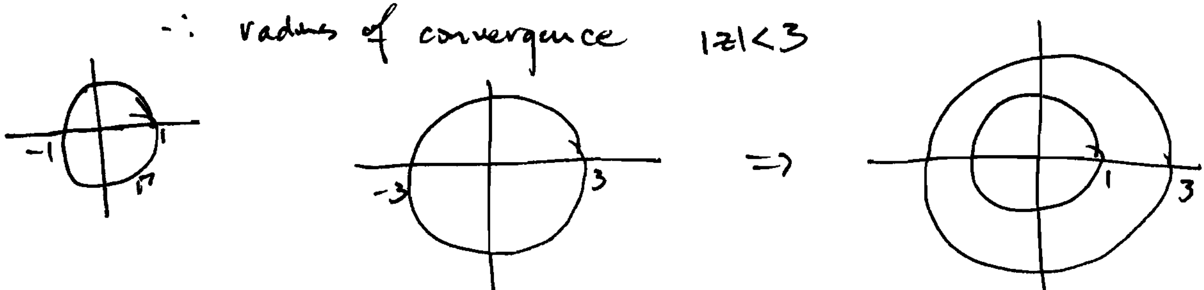
$$\lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| \Rightarrow \lim_{k\to\infty} \left| \frac{(k+1)^3 z^{k+1} 3^k}{3^{k+1} k^3 z^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{z}{3} \frac{(k^3+3k^2+3k+1)}{k^3} \right|$$

$$= \frac{z}{3} \lim_{k\to\infty} \left| \cancel{\frac{k^3}{k^3}} \Big(1 + \underbrace{\cancel{\tfrac{3}{k}}}_{0} + \underbrace{\cancel{\tfrac{3}{k^2}}}_{0} + \underbrace{\cancel{\tfrac{1}{k^3}}}_{0}\Big) \right|$$

$$= \frac{z}{3}$$

$\therefore$ radius of convergence $|z|<3$



Since around the boundary & inside $|z|=1$ is analytic also e^z is entire

$$\oint_{|z|=1} e^z f(z)\, dz = 0$$ by Cauchy's integral theorem.