#4 Find the laurent series.

$$\frac{\sin 2z}{z^3} \qquad |z| > 0$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\sin 2z = 2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots$$

putting the 2 together

$$\Rightarrow \frac{1}{z^3} \sin 2z$$

$$= \frac{1}{z^3}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)$$

$$= \frac{2}{z^2} - \frac{2^3}{3!} + \frac{2^5 z^2}{5!} - \cdots$$

$$\therefore \frac{\sin 2z}{z^3} = \sum_{k=-1}^{\infty} \frac{(-1)^{k+1}}{(2k+3)!} z^{2k} \, z^{n+1}$$

#7. $$\frac{1}{e^z - 1} \qquad 0 < |z| < 2\pi$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\frac{1}{e^z - 1} = \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}$$

$$= \frac{1}{z} \quad \frac{1}{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}$$

#7 cont.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=-\infty}^{1} a_n z^{-n}$$ ⇐ laurent Series expansion

$$= a_{-n} z^{-n} + \dots + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots$$

$$\frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots} = a_{-n} z^{-n} + \dots + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

$$1 = \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)\left(\frac{a_{-1}}{z} + a_0 + a_1 z + \dots\right)$$

$$= a_{-1} + a_0 z + a_1 z^2 + \frac{a_{-1} z}{2!} + \frac{a_0 z^2}{2!} + \frac{a_1 z^3}{2!} + \frac{z^2 a_{-1}}{3!} + \frac{a_0 z^3}{3!} +$$

$$= a_{-1} + \left(\frac{a_{-1}}{2} + a_0\right) z + \left(\frac{a_{-1}}{3!} + \frac{a_0}{2!} + a_1\right) z^2 + \dots$$

$a_{-1} = 1$

$$\frac{a_{-1}}{2} + a_0 = 0$$

$$\frac{1}{2} + a_0 = 0$$

$$a_0 = -\frac{1}{2}$$

$$\frac{a_{-1}}{6} + \frac{a_0}{2} + a_1 = 0$$

$$\frac{1}{6} + \left(-\frac{1}{4}\right) + a_1 = 0$$

$$a_1 = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

$a_{-1} = 1$

$a_1 = \frac{1}{12}$

$a_0 = -\frac{1}{2}$ $\Rightarrow$ $1 - \frac{z}{2} + \frac{z^2}{12} \dots$

SEC. 5.6

#2 $f(z) = \dfrac{1}{(2\cos z - 2 + z^2)^2}$

$\dfrac{1}{f(z)} = (2\cos z - 2 + z^2)^2$

if we expand $\cos z$

$\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dots$

$\Rightarrow \left(2\left(1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dots\right) - 2 + z^2\right)^2$

$= \left(2\left(-\dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dots\right) + z^2\right)^2$

$= 4\left(-\dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots\right)^2 + z^4 + 4z^2\left(-\dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots\right)$

$= 4\left(\dfrac{z^4}{4} - \dfrac{2z^6}{2!4!} + \dfrac{z^8}{320} - \dots\right) + z^4 - 2z^4 + \dfrac{4z^6}{4!} - \dfrac{4z^8}{6!} + \dots$

$= \left(z^4 - \dfrac{z^6}{6} + \dfrac{z^8}{80} - \dots\right) - z^4 + \dfrac{z^6}{6} - \dfrac{z^8}{180} + \dots$

$= \dfrac{z^8}{80} - \dfrac{z^8}{180} - \dots$

$= \dfrac{9z^8 - 3z^8}{720} + \dots$

$= \dfrac{6z^8}{720} + \dots$

$= \dfrac{z^8}{120} + \dots$

$= z^8\left(\dfrac{1}{120} + \dots\right)$

to flip back we have

$\dfrac{1}{z^8}(\sim)$ so pole order of 8.

#3. removable singularity @ $z=0$
pole of order 6 @ $z=1$
essential sing. @ $z=i$

pole order 6 $\Rightarrow \dfrac{1}{(z-1)^6}$

removable sing. $\Rightarrow \dfrac{\sin z}{z}$ or $\dfrac{z-\sin z}{z^2}$

essential sing. $\Rightarrow e^{\frac{1}{z-i}}$

$$\therefore f = \frac{\sin z}{z} + \frac{1}{(z-1)^6} + e^{\frac{1}{z-i}}$$

or

$$= \frac{\sin z \quad e^{\frac{1}{z-i}}}{z\,(z-1)^6}$$

either will do.

SEC 6.1

#1f. $\sin \frac{1}{3z}$

essential sing. @ $z=0$
if we do a quick substitution of $u = \frac{1}{3z}$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!}$$

$$\sin \frac{1}{3z} = \frac{1}{3z} - \frac{1}{(3z)^3 3!} + \frac{1}{(3z)^5 5!} - \cdots$$

we want

$\therefore$ Res @ $z=0 \Rightarrow \frac{1}{3}$.

#1h $\dfrac{z-1}{\sin z}$

simple pole @ $z = k\pi$ $\quad k = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \lim_{z \to k\pi} (z - k\pi) \frac{z-1}{\sin z}$$

$$= \lim_{z \to k\pi} \frac{z^2 - z - k\pi z + k\pi}{\sin z} \quad \text{use l'hospital}$$

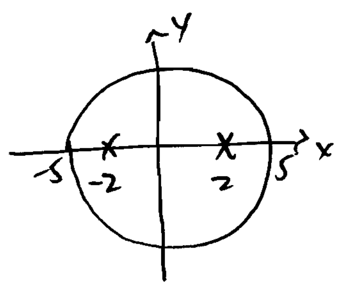
$$= \lim_{z \to k\pi} \frac{2z - 1 - k\pi}{\cos z}$$

$$= \frac{2k\pi - 1 - k\pi}{\cos k\pi} = \frac{k\pi - 1}{\cos k\pi}$$

$$\cos k\pi = \begin{cases} 1 & k = 0, 2, 4, \ldots \\ -1 & k = 1, 2, 3, \ldots \end{cases}$$

$$\therefore \Rightarrow (-1)^n (k\pi - 1)$$

#3a) $\displaystyle\oint_{|z|=5} \frac{\sin z}{z^2 - 4}\, dz$



poles @ $z = 2$ & $z = -2$

Res @ $z = 2$

$$\Rightarrow \lim_{z \to 2} (z-2) \frac{\sin z}{(z-2)(z+2)} = \lim_{z \to 2} \frac{\sin z}{z+2} = \frac{\sin 2}{4}$$

Res @ $z = -2$

$$\Rightarrow \lim_{z \to -2} (z+2) \frac{\sin z}{(z-2)(z+2)} = \lim_{z \to -2} \frac{\sin z}{z-2} = \frac{\sin(-2)}{-4} = \frac{-\sin 2}{-4} = \frac{\sin 2}{4}$$

#3a Cont.

using the residue thm

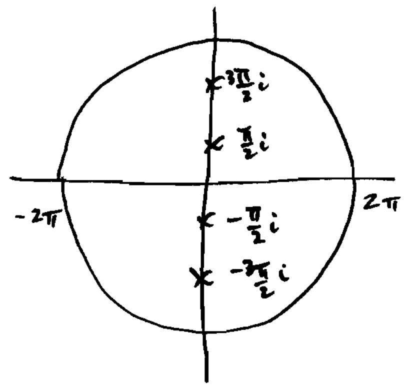
$$\oint_{|z|=5} \frac{\sin z}{z^2-4} = 2\pi i \sum Res.$$

$$= 2\pi i \left(\frac{\sin 2}{4} + \frac{\sin 2}{4} \right)$$

$$= 2\pi i \left(\frac{2 \sin 2}{4} \right)$$

$$= \pi i \sin 2$$

3c $\oint_{|z|=2\pi} \tan z \, dz$



radius $= 2\pi$

$$= \oint \frac{\sin z}{\cos z} dz$$

sing. @ $z = \frac{n\pi}{2}$ $\quad n = \pm 1, \pm 2, \pm 3, \ldots$

Since $\sin z$ & $\cos z$ is entire functions;

$$f(z) = \frac{P(z)}{Q(z)} \Rightarrow \begin{array}{l} P(z) = \sin z \\ Q(z) = \cos z \\ Q'(z) = -\sin z \neq 0 \end{array}$$

using the residue thm.

$$\oint \tan z \, dz = 2\pi i \sum Res.$$

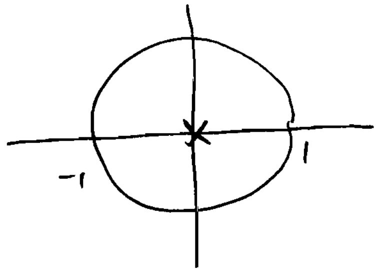
so the Res @ $z = \frac{\pi}{2}$

$\lim_{z \to \frac{\pi}{2}} (-1) = -1$ $\quad$ same with $-\frac{\pi}{2}, \pm \frac{3\pi}{2}$

$$\therefore \oint \tan z \, dz = 2\pi i \sum Res.$$

$$= 2\pi i (-1-1-1-1) = -8\pi i$$

#3e) $\oint_{|z|=1} \frac{1}{z^2 \sin z}\, dz$



pole order 3 @ $z=0$

expand $\sin z$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$f(z) = \frac{1}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)}$$

$$= \frac{1}{z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)}$$

$$= \frac{1}{z^3}\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \dots\right)$$

$$= \frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z}{5!}$$

this is ~~what~~ what we need.

$$\oint \frac{1}{\sin z\, z^2}\, dz = 2\pi i \sum \text{Res.}$$
$$= 2\pi i \left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

#7. $\oint e^{\frac{1}{z}} \sin \frac{1}{z}\, dz$

essential sing. $z=0$

let $u = \frac{1}{z}$

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \Rightarrow e^{\frac{1}{z}} = 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2\left(\frac{1}{2!}\right) + \left(\frac{1}{z}\right)^3 \frac{1}{3!} + \dots$$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots \Rightarrow \sin\frac{1}{z} = \frac{1}{z} - \left(\frac{1}{z}\right)^3 \frac{1}{3!} + \left(\frac{1}{z}\right)^5 \frac{1}{5!} - \dots$$

#7 cont.

$$f(z) = e^{\frac{1}{z}} \sin \frac{1}{z}$$

$$= \left(1 + \frac{1}{z} + \frac{1}{z^2 2!} + \cdots\right)\left(\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^2} + \frac{1}{z^3 2!} - \cdots$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{\cancel{1}'}{z^3 \cancel{6}_3}$$

$$= \frac{z+1}{z^2}$$

$\Rightarrow$ Res @ $z=0$

$$\lim_{z \to 0} \frac{d}{dz}\left[z^2 \left(\frac{z+1}{z^2}\right)\right] = 1$$

$$\therefore \oint_{|z|=1} e^{\frac{1}{z}} \sin \frac{1}{z}\, dz = 2\pi i \sum \text{Res}$$

$$= 2\pi i\,(1)$$

$$= 2\pi i$$