#3. $\displaystyle\int_0^{\pi} \frac{d\theta}{(3+2\cos\theta)^2} = \frac{3\pi\sqrt{5}}{25}$

let $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$

$\quad = iz\,d\theta$

$\dfrac{dz}{iz} = d\theta$

$\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$\quad = \dfrac{z+z^{-1}}{2} = \dfrac{z^2+1}{2z}$

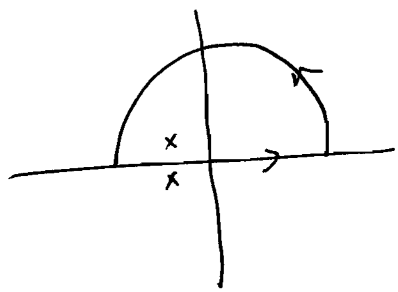
Consider

$$\Rightarrow \frac{1}{2}\int_0^{2\pi} \frac{dz}{iz}\left(\frac{1}{\left(3+2\left(\frac{z^2+1}{2z}\right)\right)^2}\right)$$

$$\Rightarrow \frac{1}{2i}\int_0^{2\pi} \frac{z\,dz}{(z^2+3z+1)^2} \Rightarrow \frac{1}{2i}\oint \frac{z\,dz}{(z^2+3z+1)^2}$$

the roots of $z^2+3z+1=0 \Rightarrow z = \dfrac{-3\pm\sqrt{9-4}}{2} = \dfrac{-3\pm\sqrt{5}}{2}$

the contour we want



so we will use only

$z = \dfrac{-3+\sqrt{5}}{2}$

the Residue @ $z = \dfrac{-3+\sqrt{5}}{2}$

$$\lim_{z\to\frac{-3+\sqrt{5}}{2}} \frac{d}{dz}\left(z+\frac{3-\sqrt{5}}{2}f(z)\right)$$

$$= \lim_{z\to\frac{-3+\sqrt{5}}{2}} \frac{d}{dz}\left(\frac{z}{\left(z+\frac{3+\sqrt{5}}{2}\right)^2}\right) = \frac{3}{5\sqrt{5}}$$

$$\therefore \frac{2}{i}\oint \frac{z\,dz}{(z^2+3z+1)^2} = 2\pi i \sum \text{Res}$$

$$= \cancel{2}\pi \cancel{i}\left(\frac{1}{\cancel{2i}}\right)\left(\frac{3}{5\sqrt{5}}\right) = \frac{3\pi\sqrt{5}}{25}$$

#6 $I = \int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \oint \frac{\left(\frac{z - \bar{z}}{2i}\right)^2}{a + b\left(\frac{z + \frac{1}{z}}{2}\right)} \cdot \frac{dz}{iz}$

$$= \frac{i}{2b} \oint \frac{(z^2 - 1)^2}{z^2\left(z^2 + 1 + 2\left(\frac{a}{b}\right)z\right)}$$

Poles @ $z = 0$ double.

$z = z_\pm = -\left(\frac{a}{b}\right) \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1}$ real since $\left(\frac{a}{b} > 1\right)$

$z_- \Rightarrow$ outside $\quad z_+ \Rightarrow$ inside

$$\Rightarrow I = \frac{i}{2b} \cdot 2\pi i \left(\operatorname{Res}(0) + \operatorname{Res}(z_+) \right)$$

$$\operatorname{Res}(0) = \frac{d}{dz}\left(\frac{(z^2 - 1)^2}{z^2 + 1 + 2\left(\frac{a}{b}\right)z} \right)\Bigg|_{z=0} = -2\left(\frac{a}{b}\right)$$

$$\operatorname{Res}(z_+) = \frac{(z_+^2 - 1)^2}{z_+^2 (z_+ - z_-)} = \frac{\left(z_+ - \frac{1}{z_+}\right)^2}{(z_+ - z_-)}$$

But $(z - z_+)(z - z_-) = z^2 + 2\left(\frac{a}{b}\right)z + 1$

$\Rightarrow z_+ z_- = 1 \Rightarrow \frac{1}{z_+} = z_-$

ie $\operatorname{Res.}(z_+) = \frac{(z_+ - z_-)^2}{(z_+ + z_-)} = 2\sqrt{\left(\frac{a}{b}\right)^2 - 1}$

$$I = -\frac{\pi}{b}\left\{ -2\left(\frac{a}{b}\right) + 2\sqrt{\left(\frac{a}{b}\right)^2 - 1} \right\}$$

$$= \frac{2\pi}{b^2}\left\{ a - \sqrt{a^2 - b^2} \right\}$$

#11.

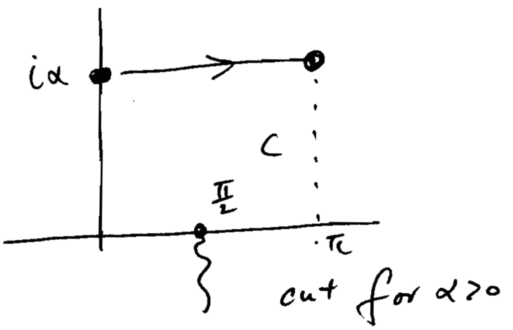
$$\int_0^{\pi} \tan(\theta + i\alpha)\, d\theta = \pi i \cdot \text{sign } a \quad a \text{ real \& nonzero}$$

$$\tan z = \frac{d}{dz} \log(\cos z) = \frac{d}{dz} F(z)$$

ie. $F(z) = \log(\cos z)$ is the antiderivative of $\tan z$

$$\Rightarrow \int_0^{\pi} \tan(\theta + i\alpha)\, d\theta = \int_C \tan z\, dz$$

$$= F(z) \Big|_{z = i\alpha}^{z = \pi + i\alpha}$$

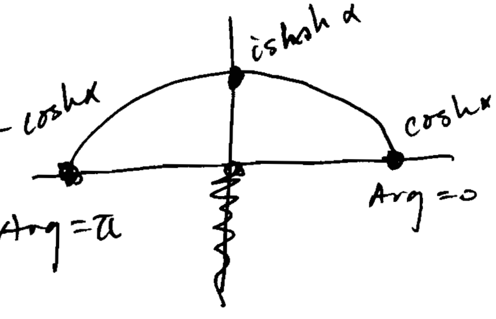


$$= \log(\cos(\pi + i\alpha)) - \log(\cos(i\alpha))$$

$\Rightarrow$ Since $\cos z$ is never zero on the line from $i\alpha$ to $\pi + i\alpha$ we can consider a branch of the log that has a cut away from the contour (Shown for $\alpha > 0$ $\rightarrow$ reverse for $\alpha < 0$)

Then $\log(\cos(\pi + i\alpha)) - \log(\cos(i\alpha))$

$$= \log\left(\frac{\cos(\pi + i\alpha)}{\cos(i\alpha)}\right) = \log\left(-\frac{\cos i\alpha}{\cos i\alpha}\right)$$



$$= \log(-1)$$

$$= i\pi \quad (-i\pi \text{ if } \alpha < 0)$$

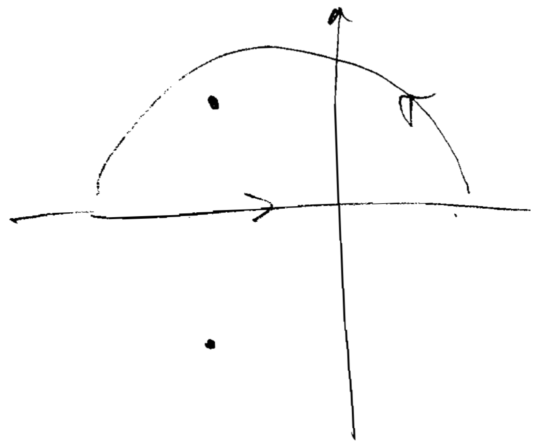
(1) $$\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \pi \qquad Q \geq P$$

$$z = \frac{-2 \pm \sqrt{4 - 4(2)}}{2}$$

$$= -1 \pm i$$

[illegible]

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2}$$



$$f(z) = \frac{1}{z^2+2z+2}$$

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2} = 2\pi i \sum \text{Res.}$$

Res @ $z = -1+i$

$$\lim_{z \to -1+i} (z+1-i) \frac{1}{\cancel{(z+1-i)}\cancel{(z+1+i)}}$$

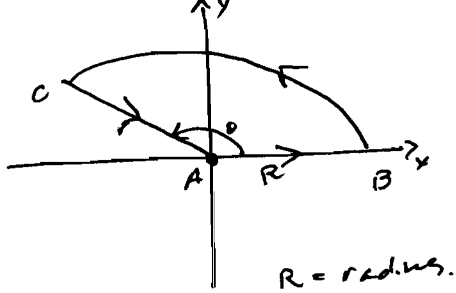
$$\lim_{z \to -1+i} \frac{1}{[(-1+i)+1+i]} = \frac{1}{2i}$$

$$\text{Res} = \frac{1}{2i}$$

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+2z+2} = 2\pi i \sum \text{Res @ } z = -1+i$$

$$= 2\pi i \left(\frac{1}{2i}\right) = \pi \checkmark$$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi\sqrt{3}}{9}$$



$\phi = \frac{2\pi}{3}$

Consider $\oint \frac{dz}{z^3+1}$

singularities occur @

$$z = e^{\frac{i\pi}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}, \ldots$$

The only ones we want are: $z = e^{\frac{\pi i}{6}}, e^{\frac{3\pi i}{6}}$

By Cauchy's Thm: $\Rightarrow \oint \frac{dz}{z^3+1} = 0$

$$\int_{AB} + \int_{BC} + \int_{CA} = 0$$

from AB $z = x$ from $x=0$ to $x=R$ $dz = dx$

BC $z = Re^{i\theta}$ " $\theta = 0$ to $\theta = \frac{2\pi}{3}$ $dz = ie^{i\theta}d\theta$

CA $z = e^{\frac{2\pi i}{3}} r$ " $r = R$ to $r = 0$ $dz = e^{2\pi i/3}dr$

$$\Rightarrow \int_{AB} \frac{dx}{x^3+1} + \int_{BC} \frac{Rie^{i\theta}\,d\theta}{(Re^{i\theta})^3+1} + \int_{CA} \frac{rie^{i\theta}\,d\theta}{(re^{i\theta})^3+1} = 0$$

~~→ 0 as R→∞ by the cauchy inequality~~

~~→ 0 as R→∞~~

The residue $\Rightarrow \frac{-2\pi i}{3} e^{\frac{\pi i}{3}}$

the 2nd integral (BC) using cauchy's inequality

$$|1+z^3| \geq |z|^n - 1 = R^n - 1$$

$$\left|\int_{BC}\right| \leq \int_{BC} \left|\frac{1}{1+z^3}\right| \leq \frac{1}{R^3-1} \int dz \to 0 \quad \text{as } R \to \infty$$

$$CA \Rightarrow \int_{CA} \frac{dz}{1+z^3} - \int_R^0 \frac{e^{\frac{2\pi i}{3}}}{1+(re^{\frac{2\pi i}{3}})^3}\,dr = -\frac{2\pi i}{3} e^{i\pi/3}$$

since $1 - e^{\frac{2\pi i}{3}} \neq 0$

#13.

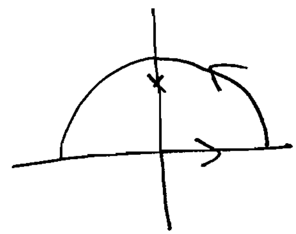
$$\Rightarrow \int_0^\infty \frac{dx}{1+x^3} = -\frac{2\pi i}{3}\left(\frac{e^{\pi i/3}}{1-e^{2\pi i/3}}\right)$$

$$= \frac{2\pi i}{3}\,\frac{1}{\sqrt{3}\,i}$$

$$= \frac{2\pi\sqrt{3}}{9} \checkmark$$

$$\Rightarrow \#13 \quad \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}dx = \frac{\pi\,(2n)!}{2^{2n}(n!)^2}$$

for $n=0,1,2,3,\ldots$



the poles $x^2+1=0 \Rightarrow x=\pm i$

the residue @ $z=i$

$$\to \lim_{z\to i} \frac{1}{n!}\frac{d^n}{dz^n}\left\{(z-i)^{n+1}f(z)\right\} = \frac{1}{n!}\,\frac{1}{n!}\,\frac{(2n)!}{2^{2n}\,i\cdot 2}$$

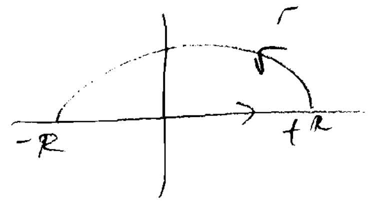
$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^{n+1}}dx = 2\pi i\sum \text{Res.}$$

$$= \frac{\pi\,(2n)!}{2^{2n}(n!)^2}$$

#2 p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(3\cos 1+\sin 1)$

consider

$$\oint_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz$$



singularities occur @ $z^2-2z+10=0$

$$z = \frac{2 \pm \sqrt{4-4(10)}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

since I have drawn the contour I will only use $z = 1+3i$

Res. @ $z = 1+3i$

$$\lim_{z \to 1+3i} \left[\cancel{(z-1-3i)} \frac{z e^{iz}}{\cancel{(z-1-3i)}(z-1+3i)} \right] = \lim_{z \to 1+3i} \frac{z e^{iz}}{z-1+3i}$$

$$= \frac{(1+3i)\left(e^{i(1+3i)}\right)}{(\cancel{1}+3i)-\cancel{1}+3i)} = \frac{1+3i}{6i}\left(e^{i}e^{-3}\right)$$

$$\oint \frac{z e^{iz}}{z^2-2z+10}\,dz = 2\pi \cancel{i}\left(\frac{1+3i}{\cancel{6}^{3}\cancel{i}}\right)\left(e^{i}\cdot e^{-3}\right)$$

$$= \frac{\pi}{3}(1+3i)\left(e^{i}\cdot e^{-3}\right)$$

Now break up the integral

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2-2x+10}\,dx + \int_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz = \frac{\pi}{3e^3}(1+3i)e^{i}$$

$$\int_{-R}^{R} \frac{x \cos x}{x^2-2x+10}\,dx + i\int_{-R}^{R} \frac{x \sin x}{x^2-2x+10}\,dx + \int_{\Gamma} \frac{z e^{iz}}{z^2-2z+10}\,dz = \frac{\pi}{3e^3}(1+3i)e$$

taking $\lim_{R\to\infty}$ the 3^{rd} integral $\to 0$

$$\Rightarrow \int_{-R}^{R} \frac{x\cos x}{x^2-2x+10}\,dx + i\int_{-R}^{R} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(1+3i)e^{i}$$

$$= \frac{\pi}{3e^3}(1+3i)(\cos 1 + i\sin 1)$$

$$= \frac{\pi}{3e^3}\cos 1 - 3\sin 1$$

$$+ \frac{\pi}{3e^3}(3i\cos 1 + i\sin 1)$$

$$\int_{-R}^{R} \frac{x\cos x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}\cos 1 - 3\sin 1$$

$$\int_{-R}^{R} \frac{x\sin x}{x^2-2x+10}\,dx = \frac{\pi}{3e^3}(3\cos 1 + \sin 1)$$

#8. $\int_0^\infty \frac{x^3 \sin 2x}{(x^2+1)^2}\,dx$

consider

$$\oint \frac{z^3 e^{2iz}}{(z^2+1)^2}\,dz$$

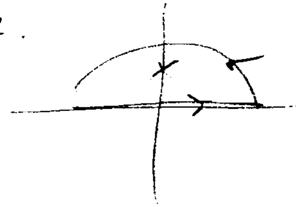
$z^2+1=0$

$z^2=-1$

$z=\pm i$ ← singularities occur

pole order 2.

use only $+i$ which is in the contour



Res. @ $z=i$

$$\lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2\; z^3 e^{2iz}}{(z-i)^2(z+i)^2}\right]$$

$$\lim_{z\to i}\frac{d}{dz}\left[\frac{z^3e^{2iz}}{(z+i)^2}\right] = \lim_{z\to i} \frac{3z^2e^{2iz}}{(z+i)^2} + \frac{2iz^3e^{2iz}}{(z+i)^2} - \frac{2z^3e^{2iz}}{(z+i)}$$

$$\Rightarrow \frac{+3e^{-2}}{+4} + \frac{2i(+i)e^{-2}}{+4} + \frac{2(-i)e^{-2}}{8i}$$

$$= \frac{3e^{-2}}{4} - \frac{2e^{-2}}{4} - \frac{e^{-2}}{4} = 0$$

$$\therefore \oint \frac{x^3\sin 2x}{(x^2+1)^2} = 0$$