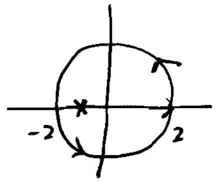
HMK #9

SEC. 4.5

$$\#3e) \int_C \frac{e^{-z}}{(z+1)^2}\,dz$$

e^{-z} is entire singularities occur @ $z=-1$ (twice)

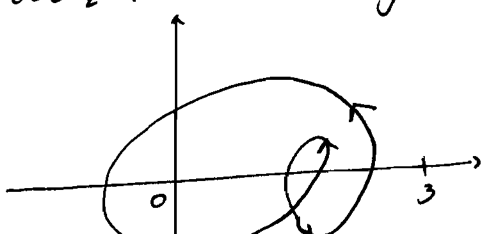


$f(z)=e^{-z}$ $f'(z)=-e^{-z}$ $z_0=-1$

$$\int_c \frac{e^{-z}}{(z+1)^2}\,dz = 2\pi i\, f'(z_0) = 2\pi i\,(-e) = -2\pi i e$$

$$\#7 \int_\Gamma \frac{\cos z}{z^2(z-3)}\,dz$$

$\cos z$ is entire singularities occur at $z=0,3$ @ zero is twice
@ 3 is once



notice that $z=0$ is in the contour
& $z=3$ is <u>not</u> in the contour

use ⤵
$\therefore z_0=0$

$$\Rightarrow \int_\Gamma \frac{\cos z}{z^2(z-3)}\,dz \qquad f(z)=\frac{\cos z}{z-3} \qquad f'(z)=\frac{-\sin z}{z-3}-\frac{\cos z}{(z-3)^2}$$

$$\int \frac{f(z)}{z^2}\,dz = 2\pi i\, f'(z_0) = 2\pi i\, f'(0) = -\frac{2\pi i}{9}$$

SEC. 4.6

$f(z) = \frac{1}{(1-z)^2}$ want to verify that $\max_{|z|=R} |f(z)| = \frac{1}{(1-R)^2}$

$$\Rightarrow f(z) = \frac{1}{(1-z)^2} \qquad z = Re^{i\theta}$$

$$|f(z)| = \left| \frac{1}{(1-z)^2} \right|$$

$$= \left| \frac{1}{(1-Re^{i\theta})^2} \right|$$

$$\max_{|z|=R} |f(z)| = \frac{1}{|(1-R)^2|} = \frac{1}{(1-R)^2} \checkmark$$

Now verify $f^{(n)}(0) = (n+1)!$

if we use $f(z) = \frac{1}{(1-z)^2}$

$$f'(z) = \frac{2}{(1-z)^3}$$

$$f''(z) = \frac{6}{(1-z)^4} = \frac{3 \cdot 2}{(1-z)^4}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}}$$

$$f^{(n+1)}(z) = \frac{(n+1)!\,(n+2)}{(1-z)^{n+3}}$$

$\therefore$ By induction $f^{(n)}(z) = \frac{(n+1)!}{(1-z)^{n+2}}$ is what we want.

$$\Rightarrow f^{(n)}(0) = \frac{(n+1)!}{1} = (n+1)! \checkmark$$

CONT.

#1. using Thm 20. on pg 215

$$f^{(n)}(z_0) \leq \frac{n!\,M}{R^n}$$

$$f^{(n)}(0) = (n+1)! \leq \frac{n!\,M}{R^n}$$

where we found M to be $\left| \frac{1}{(1-R)^2} \right|$

$$= (n+1)! \leq \frac{n!}{R^n(1-R)^2}$$ ✓

~~USE~~ NEXT PAGE

#5

let f be entire & suppose

$$\text{Re } f(z) \le M \text{ for all } z.$$

Prove that f must be a const. funct.

Since f is entire & using liouville's Thm which states that

i) $f(z)$ must be analytic

ii) $f(z)$ is bounded $f(z) \le M$ for some const.

then $f(z)$ must be a constant.

Pf. if we use cauchy's inequility

$$\left|f^{(n)}(z)\right| \le \frac{n!\,M}{R^n} \qquad \text{say } n=1$$

$$\left|f'(z)\right| \le \frac{M}{R^n}$$

let $r \to \infty$ $\quad |f'(z)| = 0 \quad$ so $f'(z) = 0$

$\therefore$ $f(z)$ must be a constant

$\Rightarrow \therefore f(z) = \text{const.}$

7.

If $f(z)$ is entire & $|f(z)| \le |z|^2$

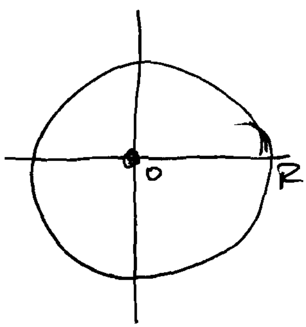
if we bound $\Rightarrow |f(z)| \le M|z|^n$ for any $|z| > r$

where n is a nonnegative #

0. Using the ~~lemma~~ Thm on pg 217

$\Rightarrow$ ~~Suppose that $f(z)$ is analytic in a disk centered at z_0 & that the maximum value of $|f(z)|$ over this disk is $|f(z_0)|$. Then $|f(z)|$ is a constant in the disk.~~

$\Rightarrow$ If f is analytic in a Domain D & $|f(z)|$ achieves its maximum value at a pt. z_0 in D, then f is constant in D.



$|z| < R$

$f(0) = i$

$|f(z)| \le 1 = M$.

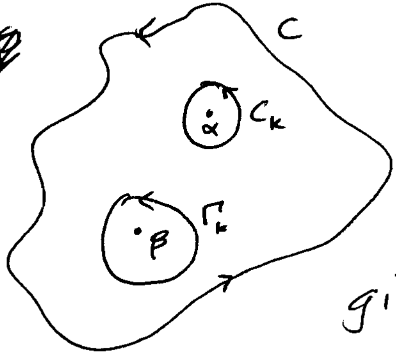
17. Find $\max\limits_{|z| \le 1} |(z-1)(z+\frac{1}{2})|$

$= |z^2 - \frac{1}{2}z - \frac{1}{2}|$

$\le |z^2| + |\frac{1}{2}z| + |\frac{1}{2}|$

≤ 2

#18.



pole $z = \alpha$ of order (multiplicity) of p

zero $z = \beta$ of order (") of n

given $\Rightarrow \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{P'(z)}{P(z)}\, dz$

$$\Rightarrow P = a(z-z_1)^{n_1} \cdots (z-z_k)^{n_k}$$

$$n_1 + \cdots + n_k = n$$

$$\frac{P'}{P} = \frac{n_1}{z-z_1} + \cdots + \frac{n_k}{z-z_k}$$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{P'(z)}{P(z)}\,dz = \sum_{r=1}^{j} \frac{1}{2\pi i}\oint_{\Gamma_r} \frac{P'(z)}{P(z)}\,dz + \sum_{r=1}^{k} \frac{1}{2\pi i}\oint_{C_r} \frac{P'(z)}{P(z)}\,dz$$

$$= \sum_{r=1}^{j} n_r - \sum_{r=1}^{k} p_r$$

$$= n - p.$$