314 '03-Section 6.2 solutions

December 6, 2003

1 $<1c>$

$$
\begin{aligned}
y_1' &= y_1 - 2y_2 \\
y_2' &= -2y_1 + 4y_2
\end{aligned}
$$

Solution

$$
\frac{d\mathbf{y}'}{dt} = A\mathbf{y} := \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to \mathbf{y} = e^{At}\mathbf{y}_0 \ = Ve^{Dt}V^{-1}\mathbf{y}_0,
$$

$$
A = VDV^{-1}\ ,\ V = [\mathbf{v}_1, \mathbf{v}_2] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
$$

$$
\begin{aligned}
A\mathbf{v}_i = \lambda_i \mathbf{v}_i \to \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} &= (1-\lambda)(4-\lambda) - (-2)(-2) = 0 \\
\lambda(\lambda - 5) = 0 &\to \lambda = 0, 5
\end{aligned}
$$

We find the eigenvectors

1. $\lambda_1 = 0$

$$
2 \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \to \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

2. $\lambda_2 = 5$

$$
-1/2 \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \to \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}
$$

Then

$$V = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \ , \ V^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \ , \ e^{Dt} = \begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix}$$

so that

$$\begin{aligned}
\mathbf{y}(t) &= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix}\frac{-1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{y}_0 \\
&= \begin{bmatrix} 2 & e^{5t} \\ 1 & 2e^{5t} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\mathbf{y}_0 \\
&= \frac{-1}{5}\begin{bmatrix} 4 - e^{5t} & -2 - 2e^{5t} \\ 2 - 2e^{5t} & -1 - 4e^{5t} \end{bmatrix}\mathbf{y}_0
\end{aligned}$$

This method finds not only the general solution but the "fundamental" solution matrix as well; from it the solution to any initial value problem is found simply by multiplying times the initial condition vector. To write a general solution, we can simply set:

$$\mathbf{y}(t) = Ve^{Dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & e^{5t} \\ 1 & 2e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

2 $< 2c >$

Solve the initial value problem:

$$\begin{aligned}
y_1' &= 2y_1 - 6y_3 \ , \ y_1(0) = 2 \\
y_2' &= y_1 - 3y_3 \ , \ y_2(0) = 2 \\
y_3' &= y_2 - 2y_3 \ , \ y_3(0) = 2
\end{aligned}$$

Solution

$$\frac{d\mathbf{y}'}{dt} = A\mathbf{y} := \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \rightarrow \mathbf{y} = e^{At}\mathbf{y}(0) \ = Ve^{Dt}V^{-1}\mathbf{y}(0),$$

$$A = VDV^{-1} \ , \ V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \to \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)(-\lambda) + (-6 + 3(2-\lambda)) = 0$$

$$\lambda\left(\lambda^2 - 4\right) + 3\lambda = \lambda\left(\lambda^2 - 1\right) = 0 \quad \to \quad \lambda = 0, \pm 1$$

We find the eigenvectors

1. $\lambda_1 = -1$

$$\begin{matrix} \\ -1/3 \\ 0 \end{matrix} \begin{bmatrix} 3 & 0 & -6 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{matrix} \\ \\ -1 \end{matrix} \begin{bmatrix} 3 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

2. $\lambda_2 = 0$

$$\begin{matrix} \\ -1/2 \\ 0 \end{matrix} \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \to \mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

3. $\lambda_3 = 1$

$$\begin{matrix} \\ -1 \\ 0 \end{matrix} \begin{bmatrix} 1 & 0 & -6 \\ 1 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{matrix} \\ \\ 1 \end{matrix} \begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

Then

$$V = \begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad V^{-1} = \frac{-1}{2} \begin{bmatrix} -1 & 3 & -3 \\ 2 & -4 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad e^{Dt} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{y}(t) &= \left(\begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix} \right) \left(\frac{-1}{2} \begin{bmatrix} -1 & 3 & -3 \\ 2 & -4 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2e^{-t} & 3 & 6e^t \\ e^{-t} & 2 & 3e^t \\ e^{-t} & 1 & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} + 6 - 6e^t \\ e^{-t} + 4 - 3e^t \\ e^{-t} + 2 - e^t \end{bmatrix} \end{aligned}$$