314, S'09-QUIZ 4

Name:____

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1 $< 12pts >$

Let $\mathbf{B}$ and $\mathbf{C}$ be fixed $n \times n$ matrices. Determine whether the following are linear operators on $\mathbf{R}^{n\times n}$.

1. (4 pts) $L(A) = C^2A + AB$. YES
2. (4 pts) $L(A) = C^2A + B$. NO: $L(0) = C^2 \cdot 0 + B = B \neq 0$
3. (4 pts) $L(A) = C^2AB$. YES

Solution

(1) $L(\alpha A_1 + \beta A_2) = C^2(\alpha A_1 + \beta A_2) + (\alpha A_1 + \beta A_2)B$
$= \alpha(C^2A_1 + A_1B) + \beta(C^2A_2 + A_2B)$
$= \alpha L(A_1) + \beta L(A_2)$

(3) $L(\alpha A_1 + \beta A_2) = C^2(\alpha A_1 + \beta A_2)B$
$= \alpha(C^2A_1B) + \beta(C^2A_2B)$
$= \alpha L(A_1) + \beta L(A_2)$

2 $< 16pts >$

Given the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbf{R}^3$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad U = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and the vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{R}^2$,

$$V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad V^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and let $\mathcal{L}$ be the linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ given by

$$(y_1, y_2)^T = \mathcal{L}(x_1, x_2, x_3) = (x_2 - x_1, x_3 - x_1)^T .$$

1. (5pts) Find a matrix $\mathbf{A}$ such that $\mathbf{y} = \mathcal{L}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ for all inputs $\mathbf{x} \in \mathbf{R}^3$ written in the standard basis $E_3 = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and giving output $\mathbf{y} \in \mathbf{R}^2$ in terms of its components in the standard basis $E_2 = (\mathbf{e}_1, \mathbf{e}_2)$.

2. (8pts) Find a matrix $\mathbf{B}$ such that $\mathcal{L}(\mathbf{x}) = \mathbf{B}\mathbf{x} = \mathbf{y}$ for all input vectors $\mathbf{x} \in \mathbf{R}^3$ written in the basis $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ and giving output $\mathbf{y} \in \mathbf{R}^2$ in terms of its components in the basis $V = (\mathbf{v}_1, \mathbf{v}_2)$.

3. (3pts) Are the matrices $\mathbf{A}$ and $\mathbf{B}$ related? How?

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "A", "\mathcal{L}"'] & \mathbb{R}^2 \arrow[d, "V^{-1}"] \\ U \arrow[u, "U"] \arrow[r, "B"] & V \end{tikzcd}$$

Solution

$$\boxed{B = V^{-1}AU}$$

(1) $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_3 - x_1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

(2-3) $B = V^{-1}AU = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -2 & 2 & -1 \\ -2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1/2 \\ -1 & 2 & 5/2 \end{pmatrix}$

$x_E = U x_u$ $\qquad A x_E = y_E \Rightarrow AUx_u = Vy_v$

$y_E = V y_v$ $\qquad \Rightarrow (V^{-1}AU)x_u = y_v$

3 < 12*pts* >

1. (2pts) Show that the following transformation from P^2 to P^3 is linear: $\mathcal{L}(p(x)) = p(1) + \int_0^x p(x)dx$.

$$\mathcal{L}(\alpha p+\beta q) = \alpha p(1)+\beta q(1) + \alpha \int_0^x p(x)dx + \beta\int_0^x q(x)dx = \alpha \mathcal{L}(p)+\beta\mathcal{L}(q)$$

2. (10pts) Write the above linear transformation in terms of a 2×3 matrix if we use the ordered basis $U = \{1-x, 1+x\}$ for P^2 and the standard ordered basis $E = \{1, x, x^2\}$ for P^3. That is, write $\mathcal{L}(\mathbf{p}) = \mathbf{q}$ as $\mathbf{A}\mathbf{p}_U = \mathbf{q}_E$ where $\mathbf{p}(x) = a_1(1-x) + a_2(1+x) \Rightarrow \mathbf{p}_U = (a_1, a_2)^T$; $\mathbf{q}(x) = b_1 + b_2x + b_3x^2 \Rightarrow \mathbf{q}_E = (b1, b_2, b_3)^T$ and

$$A\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution

$$\begin{aligned}\mathcal{L}(p) &= \mathcal{L}\left(a_1(1-x)+a_2(1+x)\right)\\ &= a_1\cdot 0 + a_2\cdot 2 + a_1\int_0^x(1-x)dx + a_2\int_0^x(1+x)dx\\ &= a_2 2 + a_1\left(x-\frac{x^2}{2}\right) + a_2\left(x+\frac{x^2}{2}\right)\\ &= (2a_2)\cdot 1 + (a_1+a_2)x + \left(-\frac{a_1}{2}+\frac{a_2}{2}\right)x^2 = b_1+b_2x+b_3x^2\end{aligned}$$

$$\Rightarrow A\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 2a_2 \\ a_1+a_2 \\ -\frac{a_1}{2}+\frac{a_2}{2} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 2 \\ 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}$$