314 Spring '09-QUIZ 5

Name:____

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1 < 10*pts* >

Let $\mathbf{x} = (1, -1, 1)^T$, $\mathbf{y} = (-1, 1, -2)^T$. Compute $||\mathbf{x} - \mathbf{y}||_1$, $||\mathbf{x} - \mathbf{y}||_2$, $||\mathbf{x} - \mathbf{y}||_\infty$. Under which norm are the two vectors closest together? Under which norm are they farthest apart?

Solution

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\3\end{pmatrix}$$

$$x - y = \begin{pmatrix}1\\-1\\1\end{pmatrix} - \begin{pmatrix}-1\\1\\-2\end{pmatrix} = \begin{pmatrix}2\\-2\\3\end{pmatrix} = z$$

$|z|_1 = |2| + |-2| + |3| = 7$ farthest

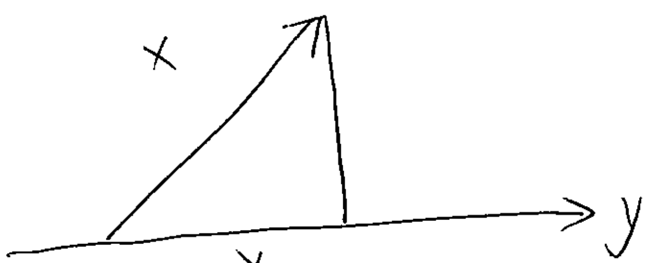
$|z|_2 = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17}$

$|z|_\infty = \max(|2|, |-2|, |3|) = 3$ closest

2 < 10*pts* >

Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$ and verify that $\mathbf{p}$ and $\mathbf{x}-\mathbf{p}$ are orthogonal if $\mathbf{x}=(1,-1,1)^T$, $\mathbf{y}=(-1,1,-2)^T$.

Solution



$$y^T x = \begin{pmatrix}-1 & 1 & -2\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix} = -1-1-2=-4$$

$$y^T y = \begin{pmatrix}-1 & 1 & -2\end{pmatrix}\begin{pmatrix}-1\\1\\-2\end{pmatrix} = 1+1+4=6$$

$$x_\parallel = \frac{y\,y^T x}{y^T y} = \qquad \frac{y^T y}{y^T y} = \frac{-4}{6} = -\frac{2}{3}$$

$$x_y = -\frac{2}{3}y = \boxed{-\frac{2}{3}\begin{pmatrix}-1\\1\\-2\end{pmatrix} = p} = \frac{1}{3}\begin{pmatrix}2\\-2\\4\end{pmatrix} = \frac{2}{3}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$x-p = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \frac{2}{3}\begin{pmatrix}-1\\1\\-2\end{pmatrix} = \begin{pmatrix}1-\frac{2}{3}\\-1+2/3\\1-4/3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

$$(\perp):\ (x-p)^T p = -\frac{2}{9}\begin{pmatrix}1 & -1 & -1\end{pmatrix}\begin{pmatrix}-1\\1\\-2\end{pmatrix} = -\frac{2}{9}(-1-1+2) = 0$$

$$R\hat{x} = \begin{pmatrix} 3/2 \\ -1/2 \\ -1/2 \end{pmatrix} \Rightarrow \begin{array}{l} 2x_1 + x_2 = 3/2 \\ x_2 - 2x_3 = -1/2 \\ 2x_3 = -1/2 \end{array}$$

Back substitution

$$2x_3 = -1/2 \Rightarrow x_3 = -1/4$$

$$x_2 = -1/2 + 2x_3 = -1$$

$$2x_1 = 3/2 - x_2 \Rightarrow x_1 = 5/4$$

$$\hat{x} = \begin{pmatrix} 5/4 \\ -1 \\ -1/4 \end{pmatrix}$$

3 $< 20pts >$

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} \qquad \hat{x} = \begin{pmatrix} 5/4 \\ -1 \\ -1/4 \end{pmatrix}$$

1. Use the Gram-Schmidt process to construct the QR factorization.
2. Use the QR factorization found in part (1) to solve the least squares problem $A\mathbf{x} = (1, -1, 0, 1)^T$.

Solution $A = QR = (q_1 q_2 q_3)\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{pmatrix} = (a_1\ a_2\ a_3)$

$$\left.\begin{array}{l} a_1 = q_1 r_{11} \\ a_2 = q_1 r_{12} + q_2 r_{22} \\ a_3 = q_1 r_{13} + q_2 r_{23} + q_3 r_{33} \end{array}\right\}$$

$$r_{11} = \|a_1\| = (1+1+1+1)^{1/2} = 2$$

$$q_1 = \frac{1}{r_{11}} a_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boxed{r_{11} = 2 \quad q_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}$$

$$r_{12} = q_1^T a_2 = \frac{1}{2}(1\ -1\ 1\ 1)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{2}{2} = 1$$

$$q_2 r_{22} = a_2 - r_{12} q_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix};$$

$$r_{22} = \|a_1 - r_{12} q_1\| = \frac{1}{2}(1+1+1+1)^{1/2} = 1$$

$$q_2 = \frac{1}{r_{22}}(a_1 - r_{12} q_1) =$$

$$\boxed{r_{12} = 1 \quad r_{22} = 1 \quad q_2 = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$r_{13} = q_1^T a_3 = \frac{1}{2}(1\ -1\ 1\ 1)\begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix} = 0 \ ; \quad r_{23} = q_2^T a_3 = \frac{1}{2}(-1\ 1\ 1\ 1)\begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix} = -\frac{4}{2} = -2$$

$$q_3 r_{33} = a_3 - q_1 r_{13} - q_2 r_{23} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ -2 \end{pmatrix} + \cancel{2} \cdot \frac{1}{\cancel{2}}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$r_{33} = (1+1+1+1)^{1/2} = 2 \ ;$$

$$\boxed{r_{13} = 0 \quad r_{23} = -2 \quad r_{33} = 2}$$

$$q_3 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\therefore \boxed{\begin{pmatrix} 1 & 0 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ & 1 & -2 \\ & & 2 \end{pmatrix}}$$

$$(2) \quad R\hat{x} = Q^T b \Rightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$$