314 '03-QUIZ 11

Name:____

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1 $< 12pts >$

Compute e^A if A is the matrix

$$A = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

Solution
We need to find eigenvalues and eigenvectors of $A = VDV^{-1}$:

$$det\,(A - \lambda I) = \begin{vmatrix} 3-\lambda & 4 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \to \lambda = \pm 1$$

1. $\lambda_1 = 1$:
$$\begin{bmatrix} 2 & 4 \\ -2 & -4 \end{bmatrix} \to \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. $\lambda_2 = -1$:
$$\begin{bmatrix} 4 & 4 \\ -2 & -2 \end{bmatrix} \to \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   Then
$$V = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \; D = \begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix}, \; V^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

   so
$$\begin{aligned} e^A &= Ve^DV^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} e^1 & e^{-1} \\ -2e^1 & -e^{-1} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -e^1 + 2e^{-1} & -e^1 + e^{-1} \\ 2e^1 - 2e^{-1} & 2e^1 - e^{-1} \end{bmatrix} \end{aligned}$$

2 $< 13pts >$

Solve the initial value problem:

$$\begin{aligned} y_1' &= 3y_1 + y_2\ ,\ y_1(0) = 1 \\ y_2' &= -4y_1 - 2y_2\ ,\ y_2(0) = 2 \end{aligned}$$

Solution

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} := \begin{bmatrix} 3 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightarrow \mathbf{y} = e^{At}\mathbf{y}_0 \ = Ve^{Dt}V^{-1}\mathbf{y}_0,$$

$$A = VDV^{-1}\ ,\ V = [\mathbf{v}_1, \mathbf{v}_2] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{v}_i = \lambda_i \mathbf{v}_i \rightarrow \begin{vmatrix} 3-\lambda & 1 \\ -4 & -2-\lambda \end{vmatrix} &= (3-\lambda)(-2-\lambda) - (1)(-4) = 0 \\ \lambda^2 - \lambda - 2 = 0 \ &\rightarrow \ (\lambda-2)(\lambda+1) = 0 \rightarrow \lambda = 2, -1 \end{aligned}$$

We find the eigenvectors

1. $\lambda_1 = 2$

$$4 \begin{bmatrix} 1 & 1 \\ -4 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. $\lambda_2 = -1$

$$1 \begin{bmatrix} 4 & 1 \\ -4 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Then

$$V = \begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix}\ ,\ V^{-1} = \frac{-1}{3} \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix}\ ,\ e^{Dt} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{y}(t) &= \left(\begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \right) \left(\frac{-1}{3} \begin{bmatrix} -4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{2t} & e^{-t} \\ -e^{2t} & -4e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{2t} - e^{-t} \\ -2e^{5t} + 4e^{-t} \end{bmatrix} \end{aligned}$$

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3 $< 12pts >$

Given the matrices A, B:

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \ , \ B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

Is there a nonsingular matrix S such that

$$S^{-1}AS = B \ ?$$

If so find it. If not, explain why not.
Solution
The problem asks for a similarity transformation between A and B. This is only possible if they have the same eigenvalues. Since B is diagonal, its eigenvalues are exhibited by its diagonal elements, i.e. they are 2 , -1. For A they need to be computed:

$$det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = 0 \rightarrow \lambda = 3 \ , \ -2$$

Therefore the two matrices have different eigenvalues, and they are not similar. No matrix S with desired properties can be found.

4 $< 13pts >$

A matrix A is normal if $AA^H = A^H A$. Show that the matrix A is normal and find an orthogonal or unitary diagonalizing matrix V such that $V^H AV = D$ with D diagonal, where

$$A = \begin{bmatrix} 0 & 2+i \\ -2+i & 0 \end{bmatrix}$$

Solution

$$A^H = \begin{bmatrix} 0 & -2-i \\ 2-i & 0 \end{bmatrix} = -A \;\rightarrow\; AA^H = A^H A = -A^2$$

so that A is normal.

$$A - \lambda I = \begin{vmatrix} -\lambda & 2+i \\ -2+i & -\lambda \end{vmatrix} = \lambda^2 - (2+i)(-2+i) = \lambda^2 + 5 = 0$$

1. $\lambda_1 = i\sqrt{5}$

$$\begin{matrix} 0 \\ \frac{1+2i}{\sqrt{5}} \end{matrix} \begin{bmatrix} -i\sqrt{5} & 2+i \\ -2+i & -i\sqrt{5} \end{bmatrix} \rightarrow \begin{bmatrix} -i\sqrt{5} & 2+i \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} 2+i \\ i\sqrt{5} \end{bmatrix}$$

2. $\lambda_2 = \; -i\sqrt{5}$

$$\begin{matrix} 0 \\ -\frac{1+2i}{\sqrt{5}} \end{matrix} \begin{bmatrix} i\sqrt{5} & 2+i \\ -2+i & i\sqrt{5} \end{bmatrix} \rightarrow \begin{bmatrix} i\sqrt{5} & 2+i \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 2+i \\ -i\sqrt{5} \end{bmatrix}$$

The two eigenvectors are orthogonal:

$$\mathbf{v}_1^H \mathbf{v}_2 = (2-i)(2+i) + (i\sqrt{5})^2 = 0$$

so, after dividing each by its magnitude we get a unitary matrix:

$$U = \left[\frac{\mathbf{v}_1}{||\mathbf{v}_1||} \,,\, \frac{\mathbf{v}_2}{||\mathbf{v}_2||} \right] = \begin{bmatrix} \frac{2+i}{\sqrt{10}} & \frac{2+i}{\sqrt{10}} \\ \frac{i\sqrt{5}}{\sqrt{10}} & -\frac{i\sqrt{5}}{\sqrt{10}} \end{bmatrix}$$