314 '03-QUIZ 7

Name:____

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1 $< 12pts >$

Given $\mathbf{u} = [1, -1, 2]^T$ and $\mathbf{w} = [2, 1, -2]^T$

1. Find the vector projection $\mathbf{p}$ of $\mathbf{u}$ onto $\mathbf{w}$.
2. Show that $\mathbf{p}$ and $\mathbf{u} - \mathbf{p}$ are orthogonal.
3. Find a vector $\mathbf{v}$ orthogonal to both $\mathbf{u}$ and $\mathbf{w}$. Do not use cross products! Show all your work.

Solution

1.

$$\mathbf{p} = \frac{1}{\mathbf{w}^T\mathbf{w}} \left(\mathbf{w}^T\mathbf{u}\right) \mathbf{w}$$

$$\mathbf{w}^T\mathbf{w} = [2, 1, -2] \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 2^2 + 1^2 + (-2)^2 = 9$$

$$\mathbf{w}^T\mathbf{u} = [2, 1, -2] \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 2 * 1 + 1 * (-1) + (-2) * 2 = -3$$

$$\mathbf{p} = \frac{-3}{9}\mathbf{w} = \frac{-3}{9}[2, 1, -2]^T$$

2.

$$\mathbf{u}-\mathbf{p} = [1,-1,2]^T + \frac{1}{3}[2,1,-2]^T = \frac{1}{3}[5,-2,4]^T$$

$$\mathbf{p}^T(\mathbf{u}-\mathbf{p}) = \frac{-1}{9}[2,1,-2]\begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} = -(2*5+1*(-2)+(-2)*4)/9 = 0$$

3. Form a matrix A with $\mathbf{u}, \mathbf{w}$ as rows. These are elements of R^3 so two of them define a plane, dimension 2. The orthogonal complement should be 1-dimensional (a line) and it forms the null space $\mathcal{N}(A)$ of A:

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \\ -2 \end{array}\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix} \rightarrow \begin{array}{c} \\ \frac{1}{3} \end{array}\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix}$$

giving

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

2 $< 13pts >$

Find a basis of the row space, column space, null space and left null space of

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

and give the dimensions of each.

Solution

$$\begin{array}{r} 1 \\ 1 \\ -2 \end{array} \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow \begin{array}{r} 0 \\ 1 \\ 1 \end{array} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & -5 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

Putting pieces together, we have shown that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

So;

$$\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\} \quad , \quad \mathcal{R}(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{N}(A^T) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \quad , \quad \mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$$