Math. 314
Set XI
Fall 09

p.133, 10(ace)
p.143, 2(ac)
p.144, 6(bc)
p.150, 7 (not assigned)

Sec. 3.2 / p.133, 10(ace) Which are spanning sets of $\mathbb{R}^3$?

(a) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\} \Rightarrow \begin{pmatrix}1&0&1\\0&1&0\\0&1&1\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ no zero rows

nonsingular matrix; independent columns $\Rightarrow$ 3 independent vectors in $\mathbb{R}^3 \Rightarrow$ a basis of $\mathbb{R}^3$.

(c) $\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}\right\} \Rightarrow \begin{pmatrix}2&3&2\\1&2&2\\-2&-2&0\end{pmatrix} \xrightarrow{\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}} \begin{pmatrix}1&2&2\\2&3&2\\-2&-2&0\end{pmatrix} \xrightarrow{\begin{pmatrix}1&&\\-2&1&\\2&0&1\end{pmatrix}} \begin{pmatrix}1&2&2\\0&-1&-2\\0&2&4\end{pmatrix} \xrightarrow{\begin{pmatrix}1&&\\0&1&\\0&2&1\end{pmatrix}} \begin{pmatrix}1&2&2\\0&-1&-2\\0&0&0\end{pmatrix}$ only two pivots $\Rightarrow$

$\Rightarrow$ only two independent columns $\Rightarrow$ not spanning set for $\mathbb{R}^3$.

(e) $\{(1,1,3)^T, (0,2,1)^T\}$: only two vectors; cannot span $\mathbb{R}^3$!

Sec. 3.3 / p.135, 2(a,c) Determine whether or not vectors independent:

(2a) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\} \to$ independent by calculation in previous problem 3.2.10(a)

(2c) $\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}\right\} \to$ dependent by 3.2.10(c).

Sec. 3.3 / p.144, 6(b,c) Determine whether or not following indep. in P_3

(b) $\{2, x^2, x, 2x+3\} \to \left\{\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}3\\2\\0\end{pmatrix}\right\}$ any 4 vectors in $\mathbb{R}^3$ (hence in P_3) must be dependent, since a basis has 3 elements. (dimension = 3).

(c) $\{x+2, x+1, x^2-1\} \to \left\{\begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} \Rightarrow \begin{pmatrix}2&1&-1\\1&1&0\\0&0&1\end{pmatrix} \to \begin{pmatrix}2&1&-1\\0&1/2&-1/2\\0&0&1\end{pmatrix}$

Full pivots $\Rightarrow$ 3 vectors are independent.

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Set XII

c. 3.3 p. 135 (8, 14)
c. 3.4, p. 141
3, 10, 14(a, b)

Sec. 3.3, p. 135 (8) Determine whether the vectors $\cos x$, 1, $\sin^2(x/2)$ are linearly independent in $C[-\pi, \pi]$.

These are differentiable functions, so we can compute the Wronskian:

$$\det\begin{pmatrix} \cos x & 1 & \sin^2(x/2) \\ -\sin x & 0 & \sin x/2 \cos x/2 \\ -\cos x & 0 & \frac{1}{2}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) \end{pmatrix} = \begin{vmatrix} \sin x & \sin x/2 \cos x/2 \\ \cos x & \frac{1}{2}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \frac{1}{2}\sin x \\ \cos x & \frac{1}{2}\cos x \end{vmatrix} = 0$$

(using identities: $2\cos\theta\sin\theta = \sin 2\theta$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$)

$\therefore$ Dependent

(14) Let A be an $m \times n$ matrix. Show that if A has linearly independent column vectors, then $N(A) = \{\underline{0}\}$

Let $A = (\underline{a}_1, \ldots, \underline{a}_n)$, with $\underline{a}_i \in \mathbb{R}^m$. Then $\underline{x} \in N(A)$

$\Rightarrow A\underline{x} = 0 \Rightarrow (\underline{a}_1 \ldots \underline{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$

$\Rightarrow x_1\underline{a}_1 + x_2\underline{a}_2 + \ldots + x_n\underline{a}_n = 0$. But if $\{\underline{a}_i\}_{i=1}^n$ are linearly independent, then the only linear combination that vanishes is the trivial combination i.e. $x_1 = x_2 = \ldots = x_n = 0$. But then, the only vector in $N(A)$ is $\underline{x} = \underline{0}$, i.e. $N(A) = \{\underline{0}\}$.

Sec. 3.4, p. 141 #3 Show that, if $\underline{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\underline{x}_3 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, then

(a) $\underline{x}_1, \underline{x}_2$ form basis of $\mathbb{R}^2$

$(\underline{x}_1, \underline{x}_2) = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$; $\det = 6 - 4 = 2 \neq 0$ nonsingular $\Rightarrow \underline{x}_1, \underline{x}_2$ independent

$\Rightarrow$ set of 2 indep vectors in $\mathbb{R}^2$ i.e. basis

(b) Any three vectors in $\mathbb{R}^2$ must be dependent (we already found a basis of size 2).

(c) Since $\underline{x}_1, \underline{x}_2$ independent, $\text{Span}(\underline{x}_1, \underline{x}_2, \underline{x}_3) = \text{Span}(\underline{x}_1, \underline{x}_2) = \mathbb{R}^2$ i.e. has dimension 2.

$\boxed{3.4.10, \text{p.}141}$ The vectors $(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \underline{x}_5) = \begin{array}{r} \\ -2 \\ -2 \end{array}\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 5 & 3 & 7 & 1 \\ 2 & 4 & 2 & 4 & 0 \end{pmatrix} \longrightarrow$

span $\mathbb{R}^3$. Pare down to a basis:

$$\longrightarrow \begin{array}{c} \begin{array}{ccccc} & & * & * & \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \end{array}$$

Since columns 1, 2, 5 correspond to basic variables, corresponding columns (i.e. $\underline{x}_1, \underline{x}_2, \underline{x}_5$) in original matrix are independent.

$\boxed{3.4.14(\text{a,b})}$ Find dimension of subspace of P_3 spanned by:

(a) $x,\ x-1,\ x^2+1 \longrightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ columns are independent, so no free variables $\Rightarrow$ dim span $= 3 \Rightarrow$ $\underline{\text{span} = P_3}$

(b) $x,\ x-1,\ x^2+1,\ x^2-1$: dimension of span is same as in part (a), since P_3 is 3-dimensional and $x, x-1, x^2+1$ are independent & span P_3. The x^2-1 must add nothing new (i.e. it must depend on first 3 - easy check).

Sec. 3.4, p.150, 7 Find a basis for

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \,\middle|\, x = a+b,\ y = a-b+2c,\ z = b,\ w = c \right\} \subset \mathbb{R}^4$$

we have
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

we need to pick a subset of independent columns:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ 3 pivots $\Rightarrow$ 3 independent columns.

$\Rightarrow$ columns of original matrix are independent $\Rightarrow$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

i.e. $S = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ (given)

and since these three vectors are independent they form a basis of S.