Sec. 3.4, p.150, 7 > Find a basis for

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \,\middle|\, x = a+b,\ y = a-b+2c,\ z = b,\ w = c \right\} \subset \mathbb{R}^4$$

we have
$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} a+b \\ a-b+2c \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

we need to pick a subset of independent columns:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ 3 pivots $\Rightarrow$ 3 independent columns.

$\Rightarrow$ columns of original matrix are independent $\Rightarrow$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

i.e. $S = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ (given)

and since these three vectors are independent they form a basis of S.

3.4.14 In each of the following, find the dimension of the subspace of P_3 spanned by the given vectors

(a) $\{x, x-1, x^2+1\} \rightarrow \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 3 pivots

$\Rightarrow$ independent

$\Rightarrow \dim(\text{span}) = 3$

(b) $\{x, x-1, x^2+1, x^2-1\} \rightarrow \begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

still 3 pivots, $\dim(\text{span}) = 3$

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Sec. 3.6 (p.159)
1(b), 2(b,c)

(1b) Find a basis for $R(A)$, $R(A^T)$, $N(A)$:

$$\begin{pmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{pmatrix} \xrightarrow{P_{12}} \begin{array}{c} \\ 3 \\ 3 \end{array}\begin{pmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{pmatrix} \to$$

$$\to \begin{array}{c} \\ \\ -2 \end{array}\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to$$

$$\to \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{1} \to \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{-2} \to \begin{array}{c} \begin{array}{cccc} & & & * \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \text{basic} \qquad \text{free} \end{array}$$

$\underline{\text{Rank} = 3}$
$\underline{\text{nullity} = 1}$

Basis of $R(A)$: $\left\{ \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} \right\}$

Basis of $N(A) = \left\{ \begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix} \right\}$ or $\begin{pmatrix} 10 \\ 2 \\ 0 \\ 7 \end{pmatrix}$ (same thing)

Basis of $R(A^T)$: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -10/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2/7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$N(A)$:
$$\begin{aligned} x_1 - 10/7\, x_4 &= 0 \\ x_2 - 2/7\, x_4 &= 0 \\ x_3 &= 0 \end{aligned}$$
let $x_4 = \alpha$: $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} 10/7 \\ 2/7 \\ 0 \\ 1 \end{pmatrix}$

(2b) Determine dimension of span: $\to$ $\underline{\text{find Rank}}$

$$\begin{array}{c} \\ -1 \\ -1 \end{array}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \to \begin{array}{c} \\ \\ -2 \end{array}\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{pmatrix}$$

3 pivots $\Rightarrow$ Rank $= 3$

$\Rightarrow$ dim span $= 3$.

(c)
$$\begin{array}{c} \\ 1 \\ -2 \end{array}\begin{pmatrix} 1 & -2 & 3 & 2 \\ -1 & 2 & -2 & -1 \\ 2 & -4 & 5 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \to \begin{array}{c} \begin{array}{cccc} & * & & * \end{array} \\ \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

2 pivots $\Rightarrow$ rank $A = 2$ $\Rightarrow$ dim span $\{\ldots\} = 2$

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Sec. 3.6, p. 159
(4d, 7ab, 9ab)

3.6.4d is $\vec b$ in column space; is system consistent?

$A x = b \longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix} \vec x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (obviously not: $\mathcal{R}(A) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \not\ni \vec b$

Augmented matrix:

$$\begin{array}{r} \\ -1 \\ -1 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow \text{inconsistent}$$

$\Longrightarrow \vec b \notin \mathcal{R}(A)$.

3.6.7ab $A^{m\times n}$, $m>n$; $\vec b \in \mathbb{R}^m$, $N(A) = \{\vec 0\}$ ($\Leftrightarrow$ A has independent columns)

(a) $A\vec x = \vec 0 \Leftrightarrow x_1 \vec a_1 + \dots + x_n \vec a_n = \vec 0$. This means that a null vector $\vec x \in \mathbb{R}^n$ implies a linear dependence condition among the columns of A. Thus, if $N(A) = \{\vec 0\}$, there is no such dependence condition, i.e. the columns of A are independent. Thus the columns of A, $\{\vec a_i\}_{i=1}^n$ are a basis of $\mathcal{R}(A)$. ~~But dim~~ Now $\mathcal{R}(A) \underset{\text{subspace}}{\subset} \mathbb{R}^m$, but $\dim \mathcal{R}(A) = n < m$.
Thus $\{\vec a_i\}_{i=1}^n$ do not span $\mathbb{R}^m$.

(b) if $\vec b \notin \mathcal{R}(A)$ then the system $A\vec x = \vec b$ is inconsistent and has no solutions.
If $\vec b \in \mathcal{R}(A)$ then the solution is unique, since there are no null vectors.

3.6.9ab (a) $A \sim B$ (we use "$\sim$" to denote "row equivalent").
Then ~~dim~~ $\mathcal{R}(A^T) = \mathcal{R}(B^T) \Rightarrow \dim \mathcal{R}(A^T) = \dim \mathcal{R}(B^T) \Rightarrow \operatorname{rank} A = \operatorname{rank} B$
$\Rightarrow \dim \mathcal{R}(A) = \dim \mathcal{R}(B)$.

(b) The only thing we know for sure is that the two spaces have the same dimension. In general they will be different: for example $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ are $A \sim B$. But $\mathcal{R}(A) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \neq \mathcal{R}(B) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.