MA514 ~~[illegible]~~

Set ~~[illegible]~~ 15

152, Sec. 3.5

1a, 5, 2a, 7, 9)

1a Find transition matrix for change from $[\underline{u}_1, \underline{u}_2] \to [\underline{e}_1, \underline{e}_2]$, $\underline{u}_1 = (1,1)^T$, $\underline{u}_2 = (-1,1)^T$

Must expres $(\underline{u}_1, \underline{u}_2)$ in terms of $(\underline{e}_1, \underline{e}_2)$:

$$\left.\begin{aligned} \underline{u}_1 &= s_{11}\underline{e}_1 + s_{21}\underline{e}_2 \\ \underline{u}_2 &= s_{12}\underline{e}_1 + s_{22}\underline{e}_2 \end{aligned}\right\} \Rightarrow \begin{pmatrix}1\\1\end{pmatrix} = s_{11}\begin{pmatrix}1\\0\end{pmatrix} + s_{21}\begin{pmatrix}0\\1\end{pmatrix}: \quad \begin{aligned} s_{11} &= 1 \\ s_{21} &= 1 \end{aligned}$$

$$\begin{pmatrix}-1\\1\end{pmatrix} = s_{12}\begin{pmatrix}1\\0\end{pmatrix} + s_{22}\begin{pmatrix}0\\1\end{pmatrix}: \quad \begin{aligned} s_{12} &= -1 \\ s_{22} &= 1 \end{aligned}$$

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\text{i.e. } (\underline{u}_1\ \underline{u}_2) = (\underline{e}_1\ \underline{e}_2)\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}\right)$$

(rule to make 1st col.) (rule to make 2nd col)

2a Find the transition matrix for the change $(\underline{e}_1, \underline{e}_2) \to (\underline{u}_1, \underline{u}_2)$ in (1a):

now the matrix is the inverse of the previous one:

$$S' = S^{-1} = \frac{1}{1\cdot 1 - (-1)\cdot 1}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

i.e.

$$\left.\begin{aligned} \underline{e}_1 &= \tfrac{1}{2}\underline{u}_1 - \tfrac{1}{2}\underline{u}_2 \\ \underline{e}_2 &= \tfrac{1}{2}\underline{u}_1 + \tfrac{1}{2}\underline{u}_2 \end{aligned}\right\} \Rightarrow \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \underline{e}_1 \quad \underline{e}_2 \end{array} = \cancel{\begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}} \begin{array}{c} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ \underline{u}_1 \quad \underline{u}_2 \end{array} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

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5 Find transition matrix for change $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \to (\underline{u}_1, \underline{u}_2, \underline{u}_3)$ if $\underline{u}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix}2\\3\\4\end{pmatrix}$

Now

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \cancel{\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

i.e. S is the inverse of the matrix w. columns $\underline{u}_1, \underline{u}_2, \underline{u}_3$

We have:

$$\begin{array}{c} \\ 1 \\ -1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \rightarrow \begin{array}{c} \\ \\ -1 \end{array}\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \rightarrow$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)\begin{array}{c} -1 \\ \\ \\ \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

$\llcorner S'$

i.e. the transition matrix from $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ to $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ is

$S' = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and the coordinate vector of the given vectors $\underline{x}_i$: $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$, $\underline{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\underline{x}_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ with respect to the basis $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$ are given by $\underline{y}_i = S' \underline{x}_i$;

i.e. $\underline{y}_1 = S \underline{x}_1 = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$ and similarly

$\underline{y}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\underline{y}_3 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$.

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7. Given $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, ~~$\underline{v}_3 = ($~~ $S = \begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix}$

Find $(\underline{w}_1, \underline{w}_2)$ so that S is trans. mat. $(\underline{w}_1, \underline{w}_2) \rightarrow (\underline{v}_1, \underline{v}_2)$

Need $(\underline{w}_1, \underline{w}_2) = (\underline{v}_1, \underline{v}_2)\begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 9 & 4 \end{pmatrix}$

9%

9) Let $[x, 1]$, $[2x-1, 2x+1]$ be ordered bases for P_2

(a) Find trans. mat: $\underset{\underline{u}_1 \quad \underline{u}_2}{[2x-1, 2x+1]} \xrightarrow{S} \underset{\underline{w}_1 \ \underline{w}_2}{[x, 1]}$

Change to vectors:

$$\left(\underset{\underline{u}_1}{\begin{pmatrix}-1\\2\end{pmatrix}}, \underset{\underline{u}_2}{\begin{pmatrix}1\\2\end{pmatrix}}\right) \longrightarrow \left(\underset{\underline{w}_1}{\begin{pmatrix}0\\1\end{pmatrix}}, \underset{\underline{w}_2}{\begin{pmatrix}1\\0\end{pmatrix}}\right)$$

i.e.
$$\begin{pmatrix}-1 & 1\\2 & 2\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}s_{11} & s_{12}\\s_{21} & s_{22}\end{pmatrix} \Rightarrow S = \begin{pmatrix}2 & 2\\-1 & 1\end{pmatrix}$$

(b) Find trans. mat: $[x, 1] \xrightarrow{S^{-1}} [2x-1, 2x+1]$

It is the inverse of the previous matrix:

$$S' = S^{-1} = \frac{1}{1\cdot 2+1\cdot 2}\begin{pmatrix}1 & -2\\1 & 2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & -2\\1 & 2\end{pmatrix}$$

$$S' = \begin{pmatrix}+1/4 & -1/2\\+1/4 & +1/2\end{pmatrix}$$

Math. 314

Set 15

sec. 4.1

p.172 (1ab, 4, 8bc)

p.184 (1ab, 2a, 3c)

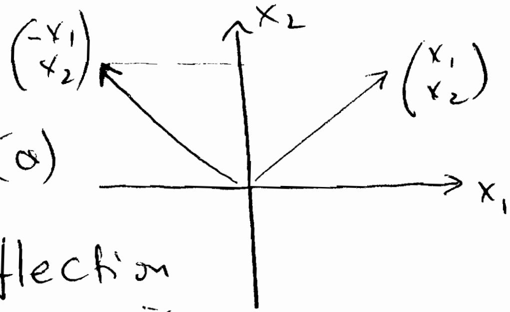
$\boxed{(172)\ 4.1.1ab}$ Show linearity, describe

(a) $L(\underline{x}) = (-x_1, x_2)^T$:

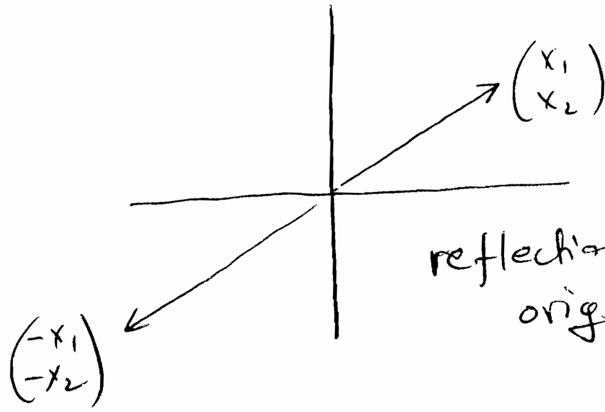
$L(\underline{x}+\underline{y}) = (-(x_1+y_1), (x_2+y_2))^T = (-x_1, x_2)^T + (-y_1, y_2)^T$

(b) $L(\underline{x}) = -\underline{x}$: $L(\underline{x}+\underline{y}) = -(\underline{x}+\underline{y}) = -\underline{x}+(-\underline{y})$

$= L(\underline{x}) + L(\underline{y})$



reflection by x_2-axis



(b)

reflection by origin

$\boxed{(172),\ 4.1.4}$ $L: \mathbb{R}^2 \to \mathbb{R}^2$ linear $\begin{cases} L\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}-2\\3\end{pmatrix} \\ L\left(\begin{pmatrix}1\\-1\end{pmatrix}\right) = \begin{pmatrix}5\\2\end{pmatrix}\end{cases}$; $L\left(\begin{pmatrix}7\\5\end{pmatrix}\right) = ?$

Since we know L is linear, $L(\alpha\underline{x}+\beta\underline{y}) = \alpha L(\underline{x}) + \beta L(\underline{y})$

So if $\underline{x} = \begin{pmatrix}1\\2\end{pmatrix}$, $\underline{y} = \begin{pmatrix}1\\-1\end{pmatrix}$, $\underline{z} = \begin{pmatrix}7\\5\end{pmatrix}$, we need to

find $\underline{z} = \alpha\underline{x} + \beta\underline{y}$ i.e.

$$\begin{pmatrix}1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}7\\5\end{pmatrix} \to {}_{-2}\left(\begin{array}{cc|c}1 & 1 & 7\\2 & -1 & 5\end{array}\right) \to \left(\begin{array}{cc|c}1 & 1 & 7\\0 & -3 & -9\end{array}\right) \to$$

$$\to \left(\begin{array}{cc|c}1 & 1 & 7\\0 & 1 & 3\end{array}\right) \to \left(\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & 3\end{array}\right)$$ i.e. $\alpha = 4, \beta = 3$ and

$$L\begin{pmatrix}7\\5\end{pmatrix} = L\left(4\begin{pmatrix}1\\2\end{pmatrix} + 3\cdot\begin{pmatrix}1\\-1\end{pmatrix}\right) = 4L\begin{pmatrix}1\\2\end{pmatrix} + 3L\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= 4\begin{pmatrix}-2\\3\end{pmatrix} + 3\begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}-8+15\\12+6\end{pmatrix} = \begin{pmatrix}+7\\18\end{pmatrix}$$

$\therefore \boxed{L\begin{pmatrix}7\\5\end{pmatrix} = \begin{pmatrix}+7\\18\end{pmatrix}}$

p.184 (4.2.1ab) Find matrices for (4.1.1ab)

(a) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$: $L(\hat{x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \hat{x}$

(b) $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$: $L(\hat{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \hat{x}$

4.2.2e $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ 0 \end{pmatrix}$$

4.2.3c $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ x_2+3x_1 \\ 2x_1-x_3 \end{pmatrix}$

$$L\hat{x} = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \hat{x}$$

4.1.8

(a) $L(A) = CA + AC$ yes

(b) $L(A) = C^2 A$ YES

(c) $L(A) = A^2 C$ NO

(c) $L(2A) = 4A^2C \neq 2L(A) = 2A^2C$

p. 173 (4.1.9 (b, c))

(b) $L(p) = x^2 + p$ NO

$L(p+q) = x^2 + p + q \neq L(p) + L(q)$

(c) $L(p) = p + xp + x^2 p'$ YES

$L(p+q) = (p+q) + x(p+q) + x^2(p'+q') = L(p) + L(q)$

$L(\alpha p) = \alpha p + x(\alpha p) + x^2(\alpha p') = \alpha L(p)$