Homework # 10
Sec. 4.2) p. 184 (4b, 5(ad), 6, 13, 14a, 15)(18ab)

Find matrix for L in standard basis.

(4b) $L(\underline{x}) = (2x_1 - x_2 - x_3,\ 2x_2 - x_1 - x_3,\ 2x_3 - x_1 - x_2)^T$

$\therefore\ L \to A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

so: $L\left(\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4-1-1 \\ -2+2-1 \\ -2-1+2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

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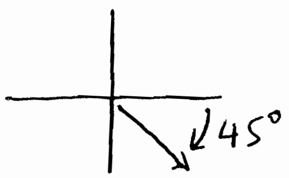
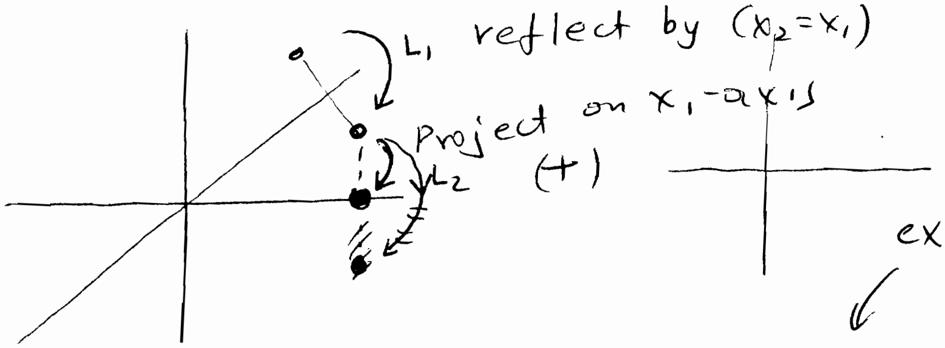
(5) Find standard matrix for: (In $\mathbb{R}^2$)

(a)



: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}_{\theta = -\frac{\pi}{4}} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

(d)



exchange (x_1, y)

$L = L_{proj} \cdot L_{refl} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

project onto x_1

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(6) $\underline{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},\ \underline{b}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\ \underline{b}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; basis $\mathbb{F}$

if $L(\underline{x}) = x_1 \underline{b}_1 + x_2 \underline{b}_2 + (x_1 + x_2)\underline{b}_3$: then

$L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_e\right) = \begin{pmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{pmatrix}_{\mathbb{F}}$: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

over

(3) $L(p(x)) = \begin{pmatrix} \int_0^1 p(x)\,dx \\ p(0) \end{pmatrix}$; $L: P_2 \longrightarrow \mathbb{R}^2$

Now, $p(x) = \alpha + \beta x \Rightarrow L(p(x)) = \begin{pmatrix} \int_0^1 (\alpha+\beta x)\,dx \\ p(0) \end{pmatrix} = \begin{pmatrix} \alpha + \frac{\beta}{2} \\ \alpha \end{pmatrix}$

$\int_0^1 (\alpha+\beta x)\,dx = \alpha x + \beta \frac{x^2}{2} \Big|_0^1 = \alpha + \frac{\beta}{2}$

$p(0) = \alpha$

So: $A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \frac{\beta}{2} \\ \alpha \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

where $p(x) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{\xi}$; $\xi = \{1, x\}$ the standard basis in P_2.

(14a) $L(p(x)) = p'(x) + p(0)$; $L: P_3 \longrightarrow P_2$

Consider the ordered bases

$\mathbb{E}: \{x^2, x, 1\}$ for P_3 and $\mathbb{F}: \{2, 1-x\}$ for P_2

Let $p(x) \in P_3$: $p(x) = \alpha x^2 + \beta x + \gamma = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}_{\mathbb{E}}$

We have:

$L(p(x)) = p'(x) + p(0) = 2\alpha x + \beta + \gamma \in P_2$

Rewrite in terms of $\mathbb{F}$: $\Big(x = -(1-x) + 1 \Big|$

$= 2\alpha[-(1-x)+1] + \gamma = -2\alpha(1-x) + 2\alpha + \gamma + \beta$

$= (-2\alpha)(1-x) + \left(\alpha + \frac{\beta+\gamma}{2}\right) \cdot 2 = \cancel{\begin{pmatrix} -2\alpha \\ \alpha + \frac{\gamma}{2} \end{pmatrix}}_{\mathbb{F}}$

$\rightarrow\rightarrow\rightarrow$

$$\Rightarrow L(p(x)) = \begin{pmatrix} \alpha + \frac{\beta+\gamma}{2} \\ -2\alpha \end{pmatrix}_F$$

$$\downarrow$$

$$A\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{pmatrix}}$$

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(13) (a): $p(x) = x^2 + 2x - 3 = \begin{pmatrix}1\\2\\-3\end{pmatrix}$

$$L(p(x)) = A\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{pmatrix}\begin{pmatrix}1\\2\\-3\end{pmatrix} = \begin{pmatrix}1/2\\-2\end{pmatrix}$$

(Indeed $p' + p(0) = 2x + 2 - 3 = 2x - 1$

$$= 2[-(1-x)+1] - 1 = -2(1-x) + 1$$

$$= \tfrac{1}{2}\cdot 2 + (-2)\cdot(1-x)$$

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(15) $S = \mathrm{span}\{e^x, xe^x, x^2e^x\} \underset{\text{subspace}}{\subset} C[a,b]$

$$\left.\begin{aligned} De^x &= e^x \\ D(xe^x) &= e^x + xe^x \\ D(x^2e^x) &= 2xe^x + x^2e^x \end{aligned}\right\}$$

Let $f \in S$;

$$f = ae^x + bxe^x + cx^2e^x = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$Df = ae^x + b(e^x + xe^x) + c(2xe^x + x^2e^x)$$

$$= (a+b)e^x + (b+2c)xe^x + cx^2e^x;$$

$$D\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a+b\\b+2c\\c\end{pmatrix}$$

$$D \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

4.2.18) Let $E = [u_1, u_2, u_3]$ and $F = [b_1, b_2]$, where

$$(\underline{u}_1, \underline{u}_2, \underline{u}_3)_E = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} = U; \quad (\underline{b}_1, \underline{b}_2)_F = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = B; \quad B^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

For each $L : L^3 \to L^2$ find matrix in (E, F):

(a) $L(x) = (x_3, x_1)^T \Rightarrow \quad A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Now in standard basis

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{S_3} \xrightarrow{L} A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{S_3} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{S_2}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{S_3} = U\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_E \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{S_2} = B\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_F$$

so $\quad AU\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_E = B\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_F \Rightarrow (B^{-1}AU)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_E = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}_F$

Matrix for L in bases (E, F) is

$$A_{E,F} = B^{-1}AU = \underbrace{\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}\overbrace{\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}}^{\begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}} = \begin{pmatrix} -1 & -3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

(b) $L(x) = (x_1 + x_2, x_1 - x_3)^T$: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = A$

Now $\quad A_{E,F} = B^{-1}AU = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -5 & -3 & 4 \\ 3 & 3 & -2 \end{pmatrix}$$