Set #21, p. 243

1c, 2, 3b, 4b

⟨5.3⟩

#21 p. 1/2

(1c) Solve by least squares:

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 4 \\ -x_1 + x_2 + x_3 &= 0 \\ -x_2 + x_3 &= 1 \\ x_1 + x_3 &= 2 \end{aligned}\right\}$$

$$\underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \underset{\hat{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} = \underset{b}{\begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}}$$

$$A^TA = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} ; \quad A^Tb = \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix}$$

Solve normal equations: $(A^TA)\hat{x} = (A^Tb)$: augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 0 & 1 & 6 \\ 0 & 3 & 1 & 3 \\ 1 & 1 & 4 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 0 & 1 & 6 \\ 0 & 3 & 1 & 3 \\ 0 & 3 & 11 & 15 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 0 & 1 & 6 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 10 & 12 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 3 & 0 & 0 & 4.8 \\ 0 & 3 & 0 & 1.8 \\ 0 & 0 & 1 & 1.2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1.6 \\ 0 & 1 & 0 & .6 \\ 0 & 0 & 1 & 1.2 \end{array}\right) : \hat{x} = \begin{pmatrix} 8/5 \\ 3/5 \\ 6/5 \end{pmatrix}$$

2 (a) $p = A\hat{x} = (17/5,\ 1/5,\ 3/5,\ 14/5)^T = \begin{pmatrix} 3.4 \\ .2 \\ .6 \\ 2.8 \end{pmatrix}$

(b) $r = b - A\hat{x} = (3/5,\ -1/5,\ 2/5,\ -4/5)^T = \begin{pmatrix} .6 \\ -.2 \\ .4 \\ -.8 \end{pmatrix}$

(c) $r \in N(A^T)$: $\boxed{A^Tr = 0}$ (check it!)

3 $A^TA = \begin{pmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{pmatrix} ; \quad A^Tb = \begin{pmatrix} 6 \\ 14 \\ 26 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}$

over →

augmented matrix : $\left(\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 6 & 14 & 26 & 26 \end{array}\right) \rightarrow \begin{array}{c} \\ \\ -3 \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 3 & 7 & 13 & 13 \end{array}\right) \rightarrow$

(obvious solution:

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$* \rightarrow$ x_3 is free

$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 7 & 7 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2^{*} & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ particular: $x_3 = 0$; $x_1 = 2$, $x_2 = 1$: $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

null vector: $x_3 = \alpha$; $x_2 = -\alpha$, $x_1 = -2\alpha$

general solution: $\boxed{\hat{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}}$

(4b) $p = A\hat{x} = \cancel{\begin{pmatrix} 6 \\ 14 \\ 26 \end{pmatrix} + \alpha \begin{pmatrix} -6 \\ -14 \\ 0 \end{pmatrix} = b ; \; p} - b = 0$

$= \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2(1-\alpha) \\ 1-\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

(notice that α does not enter: this is to be expected since A and $A^T A$ have same null space!)

$b - p = \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix}$

#22 p. 2/2

$A^T(b-p) = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

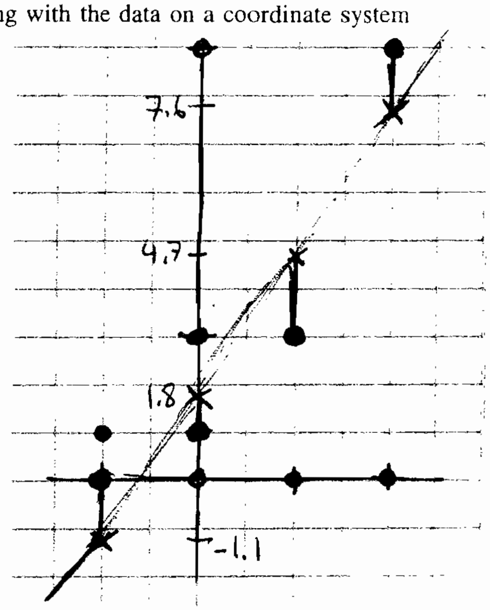
i.e. $b - p \in N(A^T)$

Math. 314
Set 20-b
(5.3 - 5, 7, 14)

5.3. 5. (a) Find the best least squares fit by a linear function to the data

x	-1	0	1	2
y	0	1	3	9

(b) Plot your linear function from part (a) along with the data on a coordinate system



(a) $$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix}$$
$$A \quad \hat{z} = b$$

$a x_i + b = y_i$

$i = 1, 2, 3, 4$

Normal equs:
$$A^T A \hat{z} = A^T b$$

$$A^T A = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$$

Then, $(A^T A)^{-1} = \frac{1}{20}\begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix}$; $\hat{z} = (A^T A)^{-1} A^T b = \begin{pmatrix} 2.9 \\ 1.8 \end{pmatrix}$

$$\boxed{y = 2.9x + 1.8}$$

5.3. 7. Given a collection of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n)^T$$
$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

and let $y = c_0 + c_1 x$ be the linear function that gives the best least squares fit to the points. Show that if $\overline{x} = 0$ then

$$c_0 = \overline{y} \quad \text{and} \quad c_1 = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{x}}$$

We have

~~$A^T A x \neq A^T$~~

$$\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$
$$A \qquad z = b$$

Then $$A^T A = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \end{pmatrix}\begin{pmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_N \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} 1 & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix}$$

and $$A^T b = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_N \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

Let $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$, $\overline{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$

while $\sum_{i=1}^{N} x_i^2 \equiv x^T x$, $\sum x_i y_i = x^T y$, so we write

$$A^T A = \begin{pmatrix} N & N\overline{x} \\ N\overline{x} & x^T x \end{pmatrix}, \quad \cancel{(A^T A)^{-1}} = \quad \text{and}$$

$$\left.\begin{aligned} \cancel{N} c_0 + \cancel{N}\overline{x} c_1 &= \cancel{N}\overline{y} \\ N\overline{x} c_0 + x^T x c_1 &= y^T x \end{aligned}\right\} \quad \text{if } \overline{x} = 0$$

Then $\underline{c_0 = \overline{y}}$, $x^T x c_1 = y^T x \Rightarrow \underline{c_1 = \frac{y^T x}{x^T x}}$

53 (14.) Find the equation of the circle that gives the best least squares circle fit to the points $(-1,-2)$, $(0, 2.4)$, $(1.1, -4)$, $(2.4, -1.6)$.

We have $(x-x_0)^2 + (y-y_0)^2 = R^2$;

x_i	-1	0	1.1	2.4
y_i	-2	2.4	-4	-1.6

where (x_0, y_0) is unknown center, R radius.

$$\Rightarrow x^2 + y^2 - 2x_0 x - 2y_0 y = R^2 - x_0^2 - y_0^2 = K.$$

Substituting given values of $x = x_i$, $y = y_i$, $i = 1,2,3,4$ get system

$$(2x_i)x_0 + (2y_i)y_0 + K = x_i^2 + y_i^2 \; ; \; i = 1,2,3,4$$

or

$$Az := \begin{pmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ 2x_3 & 2y_3 & 1 \\ 2x_4 & 2y_4 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ K \end{pmatrix} = \begin{pmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ x_3^2 + y_3^2 \\ x_4^2 + y_4^2 \end{pmatrix} = b \Rightarrow \begin{pmatrix} -2 & -4 & 1 \\ 0 & 4.8 & 1 \\ 2.2 & -8 & 1 \\ 4.8 & -3.2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ K \end{pmatrix} = \begin{pmatrix} 5.00 \\ 5.76 \\ 17.21 \\ 8.32 \end{pmatrix}$$

Normal equations $A^T A z = A^T b \Rightarrow$

$$\begin{pmatrix} 31.88 & -24.96 & 5.00 \\ -24.96 & 113.28 & -10.40 \\ 5.00 & -10.40 & 4.00 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ K \end{pmatrix} = \begin{pmatrix} 67.798 \\ -156.656 \\ 36.29 \end{pmatrix} \Rightarrow z = (A^T A)^{-1} A^T b$$

$$\Rightarrow \begin{pmatrix} x_0 \\ y_0 \\ K \end{pmatrix} = \begin{pmatrix} 57.55 \\ -64.26 \\ 668.23 \end{pmatrix} * 10^{-2} \; ; \quad R = \sqrt{K + x_0^2 + y_0^2} = 2.7252$$