[redacted], Math. 314 #21a

Sec. 5.4 (1, 3, 4a, 7b, 15a)

p. 224

$$x+y = \begin{pmatrix} 0 \\ 0 \\ 6 \\ -2 \end{pmatrix}$$

(1.) $x^T y = (-1,-1,1,1)\begin{pmatrix} 1 \\ 1 \\ 5 \\ -3 \end{pmatrix} = -1-1+5-3 = 0 \Rightarrow x \perp y$

$\|x\|_2^2 = 1+1+1+1 = 4, \quad \|y\|_2^2 = 1+1+25+9 = 36$

$\|x+y\|_2^2 = 0+0+36+4 = 40$

$\left.\right\} \|x\|_2^2 + \|y\|_2^2 = \|x+y\|_2^2$

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(3) $\underline{w} = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$; $\langle x,y\rangle_w = \sum_{i=1}^{3} w_i x_i y_i$

$= \frac{1}{4}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{4}x_3y_3$

$\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ y = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$; $\langle x,y\rangle = \frac{1}{4}(-5) + \frac{1}{2} + \frac{1}{4}\cdot 3 = 0$ (a)

(b) $\|x\|_w^2 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \Rightarrow \boxed{\|x\|_w = 1}$

$\|y\|_w^2 = \frac{1}{4}\cdot 25 + \frac{1}{2} + \frac{1}{4}\cdot 9 = \frac{36}{4} = 9 \Rightarrow \boxed{\|y\|_w = 3}$

(4a) $\langle A,B\rangle = \sum_{i,j=1}^{3} A_i B_i = 1\cdot(-4) + 2\cdot 1 + 2\cdot 1$

$+ 1\cdot(-3) + 0\cdot 3 + 2\cdot 2$

$+ 3\cdot 1 + 1\cdot(-2) + 1\cdot(-2)$

$= -4+2+2-3+4+3-2-2 = 0$

(7b) $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

Here $a=0$, $b=1$; $f = x$, $g = \sin \pi x$

$$\langle x, \sin \pi x \rangle = \int_0^1 x \sin \pi x\, dx = \frac{1}{\pi}\int_0^1 x\, d(-\cos \pi x)$$

$$= -\frac{1}{\pi} x \cos \pi x \Big|_0^1 + \frac{1}{\pi}\int_0^1 \cos \pi x\, dx$$

$$= \left(-\frac{1}{\pi} \cancel{x} \cos \pi - 0\right) + \frac{1}{\pi^2} \sin \pi x \Big|_0^1$$

$$= \frac{1}{\pi}$$

(15a) $\underline{x} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}$; $\|x\|_1 = |-3| + |4| + 0 = 7$

$$\|x\|_2 = \left((-3)^2 + 4^2 + 0^2\right)^{1/2} = 5$$

$$\|x\|_\infty = \max_{i=1,.3} |x_i| = 4$$

Set #2[redacted], due 11/10
Math. 314
Sec. 5.4, p. 224
16, 17, 24, 28, 29b

#16 $\underline{x} = (5,2,4)^T$, $y = (3,3,2)^T$

Find $\|x-y\|_i$ with $i = 1, 2, \infty$

$$x - y = (5-3,\ 2-3,\ 4-2)^T = (2,-1,2)^T$$

$\|x-y\|_1 = |2| + |-1| + |2| = 2+1+2 = 5$ farthest in 1-norm

$\|x-y\|_2 = (2^2+1^2+2^2)^{1/2} = 3$

$\|x-y\|_\infty = \max(|2|, |-1|, |2|) = 2$; closest in ∞-norm

#17 $\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x,x\rangle - \langle y,x\rangle - \langle x,y\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 - 2\langle x,y\rangle$, since $\langle x,y\rangle = 0$ if $x \perp y$,

~~when~~ we have $\boxed{x \perp y \Rightarrow \|x-y\| = \sqrt{\|x\|^2 + \|y\|^2}}$

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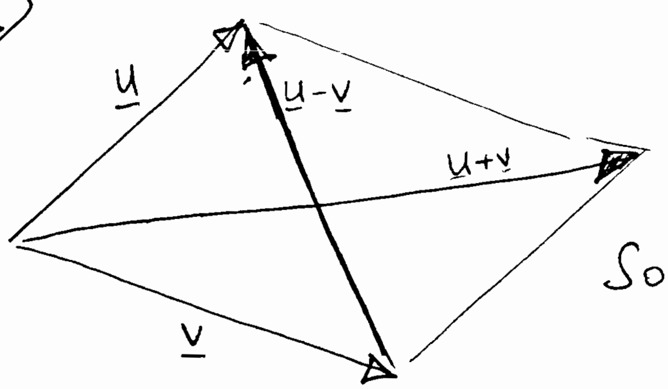


$\|u+v\|^2 = \langle u+v, u+v\rangle =$
$= \langle u,u\rangle + \langle v,v\rangle + 2\langle u,v\rangle$

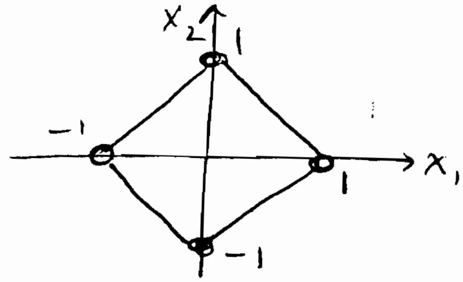
$\|u-v\|^2 = \langle u-v, u-v\rangle =$
$= \langle u,u\rangle + \langle v,v\rangle - 2\langle u,v\rangle$

So: $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

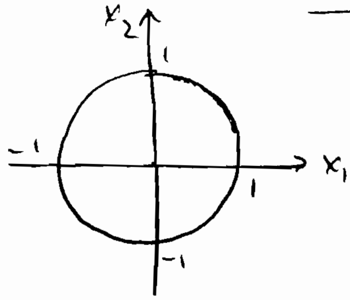
"The sums of the squares of the four sides of a parallelogram equal the sums of the squares of its two diagonals".

28) Sketch $\|\underline{x}\|_i = 1$ for $i = 1, 2, \infty$ in $\mathbb{R}^2$

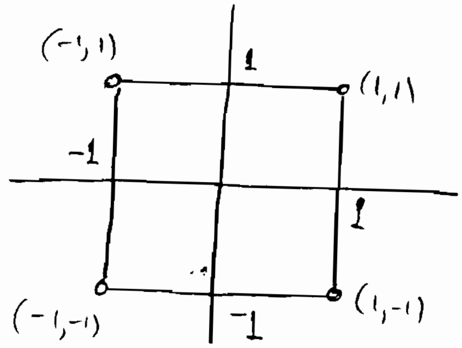
(i) $\|\underline{x}\|_1 = 1 \Rightarrow |x_1| + |x_2| = 1$



(ii) $\|\underline{x}\|_2 = 1 \Rightarrow x_1^2 + x_2^2 = 1$



(iii) $\|\underline{x}\|_\infty = 1 \Rightarrow \max(|x_1|, |x_2|) = 1$



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(29) (a) $\langle Ax, y\rangle = (Ax)^T y = x^T A^T y = x^T (A^T y)$

$= \langle x, A^T y\rangle$

$\left\{ \begin{array}{l} \langle x, y\rangle = x^T y \\ (Ax)^T = x^T A^T \\ \|x\|^2 = \langle x, x\rangle \end{array} \right.$

(b)

$$\langle A^T A x, x\rangle \overset{?}{=} \|Ax\|^2$$

$$\langle A^T A x, x\rangle = (A^T A x)^T x = x^T A^T A x$$

$$\|Ax\|^2 = \langle Ax, Ax\rangle = (Ax)^T (Ax) = (x^T A^T)(Ax)$$

$$x^T A^T A x = (x^T A^T)(Ax)$$

Solutions #22
Math. 314
5.5($2^*, 3^*, 8^*, 14^*$, ~~18~~) 21
p. 254

$\underline{5.5.2\rangle}$ Let

$$u_1 = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}, \quad u_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(a) Show $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ o.n. basis of $\mathbb{R}^3$:

Form $Q = (u_1\, u_2\, u_3)$; then show $Q^TQ = I$. Compute all pairwise dot products:

$$\langle u_1, u_1\rangle = \frac{1}{18}(1+1+16) = 1, \quad \langle u_2, u_2\rangle = \frac{1}{9}(4+4+1) = 1, \quad \langle u_3, u_3\rangle = \frac{1}{2}(1+1) =$$

$$\langle u_1, u_2\rangle = \frac{1}{9\sqrt{2}}(2+2-4) = 0, \quad \langle u_1, u_3\rangle = \frac{1}{6}(1-1) = 0, \quad \langle u_2, u_3\rangle = \frac{1}{3\sqrt{2}}\underbrace{(2-2)}_{0}$$

(b) Let $\underline{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \langle \underline{x}, \underline{u}_1\rangle \underline{u}_1 + \langle \underline{x}, \underline{u}_2\rangle u_2 + \langle \underline{x}, u_3\rangle u_3$

$$\langle x, u_1\rangle = \frac{1}{3\sqrt{2}}(1+1-4) = -\frac{\sqrt{2}}{3}; \quad \langle x, u_2\rangle = \frac{1}{3}(2+2+1) = \frac{5}{3}; \quad \langle x, u_3\rangle = \frac{1}{\sqrt{2}}(1-1) = 0$$

$$\|x\|^2 = 1+1+1 = 3 = (\langle x, u_1\rangle)^2 + (\langle x, u_2\rangle)^2 = \frac{2}{9} + \frac{25}{9} = \frac{27}{9} = 3.$$

$\underline{5.5.3\rangle}$ Let S be the subspace of $\mathbb{R}^3$ spanned by the vectors $\underline{u}_2$ & $\underline{u}_3$ of prob. (5.5.2). Let $\underline{x} = (1,2,2)^T$. Find the projection p of $\underline{x}$ onto S. Show that $(p-x) \perp u_2$ and $(p-x) \perp u_3$.

Since $u_2 \perp u_3$, and $\|u_2\| = \|u_3\| = 1$:

$$p = \langle x, u_2\rangle u_2 + \langle x, u_3\rangle u_3 = \frac{1}{3}(2+4+2)u_2 + \frac{1}{\sqrt{2}}(1-2)u_3$$

$$= \frac{8}{3}\underline{u}_2 - \frac{1}{\sqrt{2}}\underline{u}_3; \quad \text{~~Now p-x =~~} \quad p = \frac{1}{18}\begin{pmatrix} 23 \\ 41 \\ 16 \end{pmatrix}; \quad x - p = \frac{1}{18}\begin{pmatrix} -5 \\ -5 \\ 20 \end{pmatrix}$$

Now $x = \langle x, u_1\rangle u_1 + \langle x, u_2\rangle u_2 + \langle x, u_3\rangle u_3$; $\langle x, u_1\rangle = \frac{1}{3\sqrt{2}}(-5)$

So $x - p = \langle x, u_1\rangle u_1 = -\frac{5}{3\sqrt{2}}\underline{u}_1$ and by previous problem $\underline{u}_1 \perp \underline{u}_2$, $\underline{u}_1 \perp \underline{u}_3$.

5.5.8 > The functions $\cos x$ & $\sin x$ form an o.n. set in $C[-\pi,\pi]$. If

$$f(x) = 3\cos x + 2\sin x \quad \text{and} \quad g(x) = \cos x - \sin x$$

find $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx$ using the property (p. 240):

/// $\{u_i\}_{i=1}^{n}$ o.n. basis ; $v = \sum_{i=1}^{n} a_i u_i$, $w = \sum_{1}^{n} b_i u_i \Rightarrow \langle v,w\rangle = \sum_{i=1}^{n} a_i b_i$ ///

So: $\langle f,g\rangle = 3\cdot 1 - 2\cdot 1 = 1$

5.5.14 > Let u be a unit vector in $\mathbb{R}^n$ and let $H = I - 2uu^T$. Show that H is both orthogonal & symmetric and hence $H = H^{-1}$:

(i) $H^T = I - 2(uu^T)^T = I - 2(uu^T) = H$

(ii) $H^T H = (I - 2uu^T)(I - 2uu^T) = I - 4uu^T + 4u\underbrace{(u^Tu)}_{=1}u^T = I$

5.5.18 > Let

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$$A = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(a) Show that the columns of A are o.n. in $\mathbb{R}^4$

(b) Solve the l.s. problem $Ax = b$ for:

(i) $\underline{b} = (4,0,0,0)^T$; (ii) $\underline{b} = (1,2,3,4)^T$; (iii) $\underline{b} = (1,1,2,2)^T$

$$A^TA = \frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = I_2 \; ; \quad \text{then} \quad A\hat{x} = b \Rightarrow \boxed{\hat{x} = A^Tb}$$

(i) $\hat{x} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

(ii) $\hat{x} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(iii) $\hat{x} = A^T\begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+1+2+2 \\ -1-1+2+2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

5.5.28 Consider the inner product space $C[0,1]$ with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)\, g(x)\, dx$$

Let S by the subspace spanned by 1 and $2x-1$.

(a) Show 1 and $2x-1$ orthogonal

(b) Determine $\|1\|$ and $\|2x-1\|$

(c) Find the l.s. approximation to $\sqrt{x}$ by a function from S.

5.5.26 Let A be an $m\times n$ matrix whose column vectors are mutually orthogonal, and let $b \in \mathbb{R}^n$. Show that if y is the least squares solution to the system $A\underline{x} = \underline{b}$ then

$$\cancel{y}.\ y_i = a_i^T b / a_i^T a_i, \quad i = 1, \dots, n$$

A Normal equations: $A^TA\,y = A^Tb$ i.e

$$\begin{pmatrix} a_1^T \\ \vdots \\ a_i^T \\ \vdots \\ a_n^T \end{pmatrix} \begin{pmatrix} a_1 \cdots a_j \cdots a_n \end{pmatrix} y = \begin{pmatrix} a_1^T \\ \vdots \\ a_i^T \\ \vdots \\ a_n^T \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix}$$

but $a_i^T a_j = \begin{cases} \|a_i\|^2 & i=j \\ 0, & i \neq j \end{cases}$, so $A^TA = (a_i^T a_j) = \cancel{I^{n\times n}}\ \mathrm{diag}(\|a_i\|^2)$ $\quad$ $\underset{\|}{a_i^Ta_i}$

$$\begin{pmatrix} a_1^Ta_1 & & & \\ & \ddots & & 0 \\ & & a_i^Ta_i & \\ 0 & & & \ddots \\ & & & a_n^Ta_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_1^Tb \\ \vdots \\ a_i^Tb \\ \vdots \\ a_n^Tb \end{pmatrix} \Rightarrow (a_i^Ta_i)y_i = a_i^Tb,\ i=1,\dots,n$$

$$\Rightarrow y_i = a_i^Tb / a_i^Ta_i,\ i=1,\dots,n \quad D$$

~~5.5.19 Show that if U~~