Math. 314 #23
p. 270, Sec. 5.5
19, 24, 25, 26, 28

5.5.19 ~~In $\mathbb{R}^n$ with inner product $\langle x,y\rangle = x^T y$ derive a formula for the distance between two vectors~~ Show that if U is $n\times n$ orthogonal then $u_1 u_1^T + \cdots + u_n u_n^T = I$

$$U^T U = \begin{pmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{pmatrix} (u_1, u_2 \cdots u_n) = (u_i^T u_j) = I = UU^T$$

But
$$UU^T = \left[(u_1 0 \cdots 0) + (0\, u_2\, 0 \cdots 0) + \cdots + (0 \cdots 0\, u_n)\right]\left[\begin{pmatrix} u_1^T \\ 0 \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ u_n^T \end{pmatrix}\right]$$
$$= (u_1\, 0 \cdots 0)\begin{pmatrix} u_1^T \\ 0 \\ 0 \end{pmatrix} + (0\, u_2\, 0 \cdots 0)\begin{pmatrix} 0 \\ u_2^T \\ 0 \\ 0 \end{pmatrix} + \cdots$$
$$= u_1 u_1^T + \cdots + u_n u_n^T \quad \text{(all other products are } 0\text{)}.$$

5.5.24 Let A be $m\times n$, P projector from $\mathbb{R}^m$ onto $R(A)$; let Q be projector from $\mathbb{R}^n$ onto $R(A^T)$. Show that
(a) $I-P$ projects $\mathbb{R}^m$ onto $N(A^T)$
(b) $I-Q$ " $\mathbb{R}^n$ onto $N(A)$

(a) let $x \in \mathbb{R}^m$. $Px \in R(A) \Rightarrow (I-P)x \perp x \Rightarrow (I-P)x \in N(A^T)$
(b) $x \in \mathbb{R}^n$: $Qx \in R(A^T) \Rightarrow (I-Q)x \perp x \Rightarrow (I-Q)x \in N(A)$

5.5.25 Let P projector to $S \subset \mathbb{R}^m$. Show:
(a) $P^2 = P$ (b) $P^T = P$

Let $A = (a_1 \cdots a_k)$ basis of S. Then $P_s = A(A^T A)^{-1} A^T$
and $P_s^2 = [A(A^TA)^{-1}A^T][A(A^TA)^{-1}A^T] = A(A^TA)A^T = P_s$

(b) $P^T = [A(A^TA)^{-1}A^T]^T = (A^T)^T[(A^TA)^{-1}]^T A^T = A(A^TA)^{-1}A^T$
(since A^TA is symmetric, so is $(A^TA)^{-1}$)

14, 19, 24, 25
2 28 21 272

Solutions, Set #23
Math. 314, p.256
5.5 (27*, 29, 30)

5.5.27) Given the vector space $C[-1,1]$ with inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and norm $\|f\| = (\langle f,f\rangle)^{1/2}$

(a) Show that the vectors 1 and x are orthogonal

(b) Compute $\|1\|$ and $\|x\|$.

$$\langle 1,1\rangle = \int_{-1}^{1} dx = 2, \quad \langle x,x\rangle = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

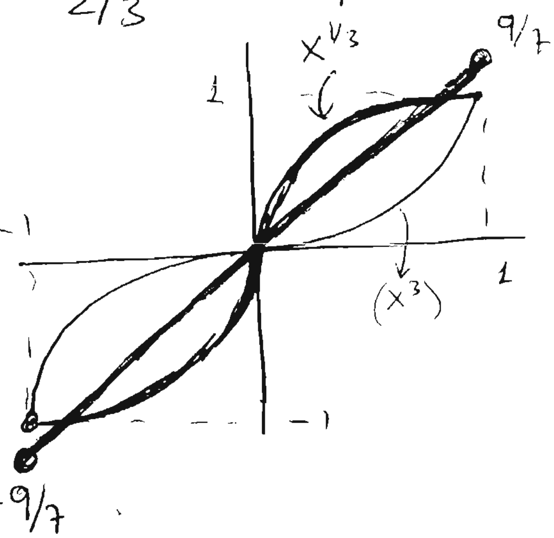
$$\langle x,1\rangle = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$$

(c) Find the best l.s. approximation to $x^{1/3}$ on $[-1,1]$ by a linear function $l(x) = c_1 1 + c_2 x$

$$\widehat{x^{1/3}} = \frac{\langle x^{1/3},1\rangle}{2}\cdot 1 + \frac{\langle x^{1/3},x\rangle}{2/3}\,x = \frac{6/7}{2/3}\,x = \frac{9}{7}x$$

$$\langle x^{1/3},1\rangle = \int_{-1}^{1} x^{1/3}\,dx = \frac{3}{4}x^{4/3}\Big|_{-1}^{1} = 0$$

$$\langle x^{1/3},x\rangle = \int_{-1}^{1} x^{4/3}\,dx = \frac{3}{7}x^{7/3}\Big|_{-1}^{1} = \frac{6}{7}$$



5.5.29) Let $S = \{1/\sqrt{2}, \cos x, \cos 2x, \dots, \cos nx, \sin x, \dots, \sin nx\}$
Show S is o.n. set with inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$.

$$\langle \cos nx, \cos mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \tfrac{1}{2}\{\cos(n+m)x + \cos(n-m)x\}\,dx = 0, \quad n \neq m$$
(true even if $n=0$)
$$= 1, \quad n = m$$

similarly for $\langle \sin nx, \sin mx\rangle = \delta_{nm}$

$$\langle \sin nx, \cos mx\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\sin(n+m)x + \sin(n-m)x\}\,dx = 0, \text{ all } n,m$$

$$\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} dx = 1$$

5.5.30 > Find the best l.s. approximation to $f(x) = |x|$ on $[-\pi, \pi]$ by a trig. poly. of deg. ≤ 2.

i.e. use basis $\left\{ \frac{1}{\sqrt{2}}, \cos x, \cos 2x, \sin x, \sin 2x \right\}$

Since $|x|$ is even, no odd components $(\langle |x|, \sin kx \rangle =$

$$\left\langle |x|, \tfrac{1}{\sqrt{2}} \right\rangle = \frac{2}{\pi} \int_0^\pi \frac{1}{\sqrt{2}} x \, dx = \frac{1}{\sqrt{2}} \frac{1}{\pi} x^2 \Big|_0^\pi = \frac{\pi}{\sqrt{2}}$$

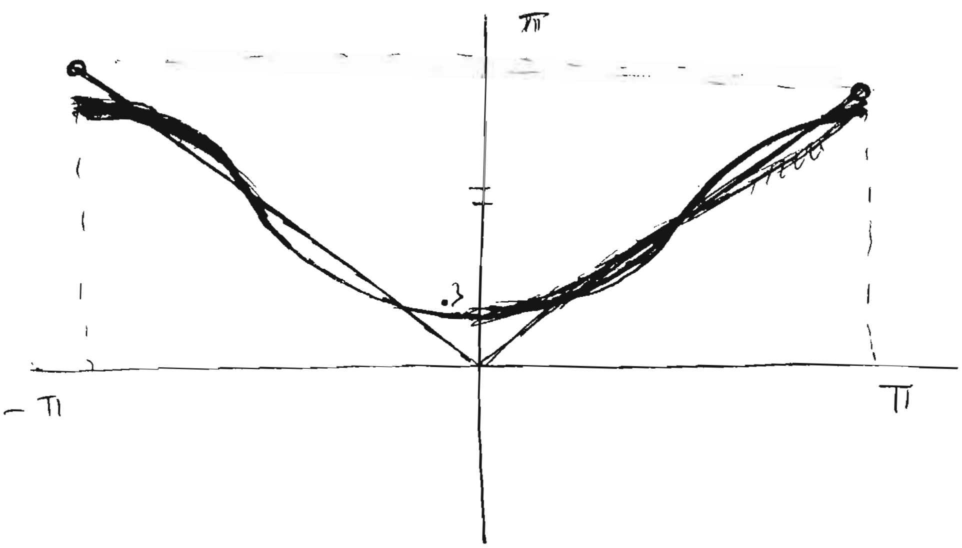
$$\langle |x|, \cos x \rangle = \frac{2}{\pi} \int_0^\pi x \cos x \, dx = \frac{2}{\pi} \int_0^\pi x \, d(\sin x) =$$

$$= \underbrace{\frac{2}{\pi} x \sin x \Big|_0^\pi}_{=0} - \frac{2}{\pi} \int_0^\pi \sin x \, dx = \frac{2}{\pi} \cos x \Big|_0^\pi = -\frac{4}{\pi}$$

$$\langle |x|, \cos 2x \rangle = \frac{2}{\pi} \int_0^\pi x \cos 2x \, dx = \frac{1}{\pi} \int_0^\pi x \, d(\sin 2x) =$$

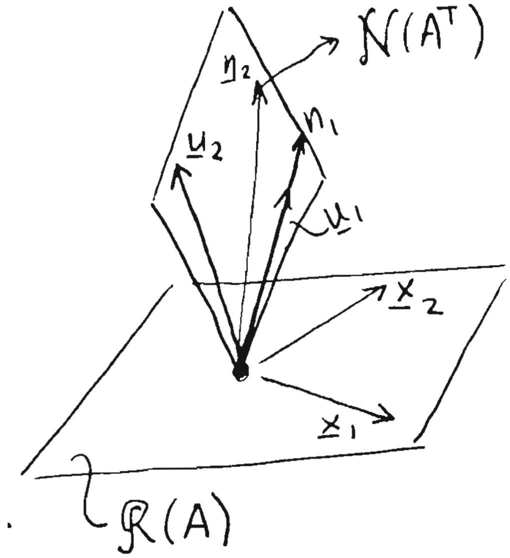
$$= \underbrace{\frac{1}{\pi} x \sin 2x \Big|_0^\pi}_{=0} - \frac{1}{\pi} \int_0^\pi \sin 2x \, dx = 0$$

i.e. $\widehat{|x|} = \frac{\pi}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \frac{4}{\pi} \cos x = \frac{\pi}{2} - \frac{4}{\pi} \cos x$



5.6.7 > The vectors $\underline{x}_1 = \frac{1}{2}(1,1,1,-1)^T$ and $x_2 = \frac{1}{6}(1,1,3,5)^T$ form an orthonormal set in $\mathbb{R}^4$. Extend this set to an o.n. basis for $\mathbb{R}^4$ by finding an o.n. basis for the nullspace of

$$A^T = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix}$$



* Given that $R(A) \oplus N(A^T) = \mathbb{R}^4$ and $R(A)$, $N(A^T)$ are orthogonal complements.
* Given that $\{\underline{x}_1, \underline{x}_2\}$ form o.n. basis of $R(A)$.
* We need an o.n. basis for $N(A^T)$.

By combining it with $\{\underline{x}_1, \underline{x}_2\}$, we will have o.n. basis of $\mathbb{R}^4$.

We first find a basis for $N(A^T)$. By its nature, it will be $\perp$ to $\underline{x}_1, \underline{x}_2$. We then apply G.S process.

(i) $A^T \underline{n} = 0 \Rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \overset{*}{1} & 1 & \overset{*}{-1} \\ 0 & 0 & 2 & 6 \end{pmatrix}$ (x_2, x_4 free)

let $x_2 = 1, x_4 = 0 \Rightarrow x_3 = 0, \; x_1 = -1$ | let $x_2 = 0, \; x_4 = 1 \Rightarrow x_3 = -3, \; x_1 = 4$

$\underline{n}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $\underline{n}_2 = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$

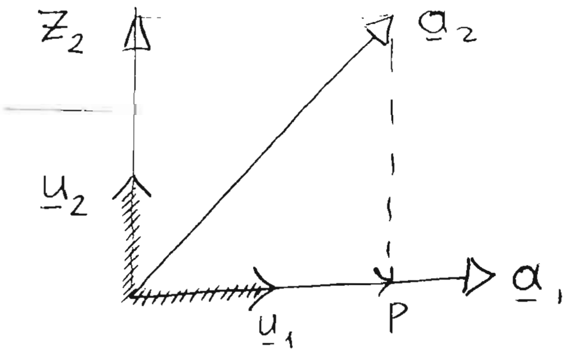
$(r_{11} = \sqrt{2})$

(ii) $\boxed{\underline{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}$; $r_{12} = \langle \underline{n}_2, \underline{u}_1 \rangle = \frac{1}{\sqrt{2}}(-4)$; $\underline{p} = r_{12} \underline{u}_1 = -2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{z}_2 = \underline{n}_2 - \underline{p} = \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}$; $r_{22} = \sqrt{4+4+9+1} = \sqrt{18}$

$\boxed{\underline{u}_2 = \frac{1}{\sqrt{18}} \begin{pmatrix} 2 \\ 2 \\ -3 \\ 1 \end{pmatrix}}$

Solutions - Set # 24
Math. 314, p. 265
5.6 (1a, 2a, 5, 7)



5.6.1a Use the Gram-Schmidt process to find o.n. basis for $R(A)$ if $A = \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}$

$$r_{11} = \|a_1\| = \sqrt{1+1} = \sqrt{2} \qquad r_{12} = \frac{1}{\sqrt{2}}(-3+5) = \sqrt{2}$$

$$u_1 = \frac{1}{r_{11}} a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$p = \underbrace{\langle a_2, u_1 \rangle}_{= r_{12}} u_1 = \frac{(-3+5)}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$z_2 = a_2 - p = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$r_{22} = \|z_2\| = 4\sqrt{2}\,; \qquad u_2 = \frac{1}{r_{22}} z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

5.6.2a Factor A above into a product QR, where Q is orthogonal and R is upper triangular

We found

$$\left. \begin{aligned} u_1 &= \frac{1}{r_{11}} a_1 \\ u_2 &= \frac{1}{r_{22}} (a_2 - r_{12} u_1) \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} a_1 &= u_1 r_{11} \\ a_2 &= u_1 r_{12} + u_2 r_{22} \end{aligned} \right\}$$

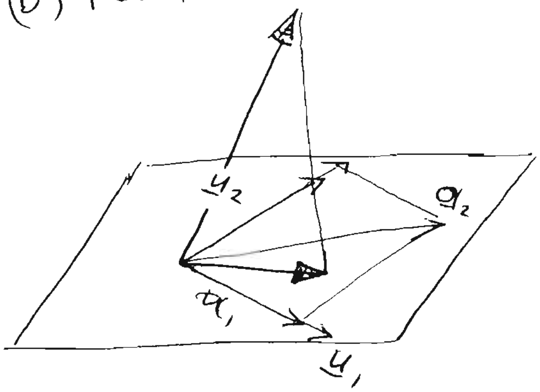
In matrix form $\quad (a_1 \ a_2) = (u_1 \ u_2) \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$

or $\quad \begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{pmatrix}$

5.6.5 Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$

(a) Use GS to find o.n. basis for $\mathcal{R}(A)$

(b) Factor A into QR.



$$r_{11} = (\langle a_1, a_1 \rangle)^{1/2} = \sqrt{4+1+4} = 3$$

$$\underline{u}_1 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$r_{12} = \langle a_2, u_1 \rangle = \frac{1}{3}(2+1+2) = 5/3$$

$$p = r_{12}\underline{u}_1 = \frac{5}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

$$z = a_2 - r_{12}\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 10/9 \\ 5/9 \\ 10/9 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

$$r_{22} = \|z\| = \frac{1}{9}\sqrt{1+16+1} = \sqrt{\frac{2}{9}}$$

$$\underline{u}_2 = \frac{1}{\sqrt{18}}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}$$

Thus $A = QR$: $\cancel{\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{18} & 2/3 \\ 4/\sqrt{18} & 1/3 \\ -1/\sqrt{18} & 2/3 \end{pmatrix}\begin{pmatrix} 3 \end{pmatrix}}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \end{pmatrix}}_{\text{o.n. basis for } \mathcal{R}(A)} \begin{pmatrix} 3 & 5/3 \\ 0 & \sqrt{18}/9 \end{pmatrix}$$

(c) Solve l.s. problem

$$Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^T b$$

$$Q^T b = \begin{pmatrix} 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \end{pmatrix}^T \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(24+6+36) \\ \frac{1}{\sqrt{18}}(-12+24-18) \end{pmatrix} = \begin{pmatrix} 22 \\ -6/\sqrt{18} \end{pmatrix}$$

so: $3x_1 + 5/3\, x_2 = 22 \Rightarrow x_1 = 27/3$

$\sqrt{18}/9\, x_2 = -6/\sqrt{18} \Rightarrow x_2 = -3$

$$\hat{x} = \begin{pmatrix} 9 \\ -3 \end{pmatrix}$$