Solutions #25
Math. 314, p. 289
6.1 (1(a,f,g,h,j), 2, 10, 13)

1. Find the eigenvalues / eigenspaces for

(a) $\begin{vmatrix} 3-r & 2 \\ 4 & 1-r \end{vmatrix} = (r-3)(r-1)-8 = r^2-4r-5 = 0$

$r = \quad (r-5)(r+1)$

$r=5: \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \rightarrow \hat{e}_5 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$r=-1: \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \hat{e}_{-1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(f) $\begin{vmatrix} -r & 1 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r \end{vmatrix} = -r^3 = 0 \Rightarrow r = 0$, triple

$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \hat{e}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, only evector

---

(g) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$: $r_1 = 1$ double; $r = 2$, simple

$(r-1)^2(r-2) = 0$

$r=1: \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$: $\hat{e}_{1,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ or $\hat{e}_{1,2} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$r=2: \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \hat{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(h) $\det\begin{pmatrix} 1-r & 2 & 1 \\ 0 & 3-r & 1 \\ 0 & 5 & -1-r \end{pmatrix} = (1-r)\left[(3-r)(-1-r)-5\right] = (1-r)\underbrace{\left(r^2-2r-8\right)}_{(r-4)(r+2)}$ $r = 1, 4, -2$

$r=-2; \begin{pmatrix} 3 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 2 & 1^{*} \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$: $\hat{e}_{-2} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$

$r=1: \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -4.5 \end{pmatrix}, \quad \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

6.1.1 h (continued)

$$r = 4, \quad \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{pmatrix} \to \begin{pmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} ; \quad \hat{e}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(j)
$$\begin{vmatrix} -2-r & 0 & 1 \\ 1 & -r & -1 \\ 0 & 1 & -1-r \end{vmatrix} = -(r+2)\begin{vmatrix} -r & -1 \\ 1 & -1-r \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -r \\ 0 & 1 \end{vmatrix}$$
$$= -(r+2)[(r+1)r+1] + 1 = -(r^3+3r^2+3r+1) = -(r+1)^3$$

$r = -1$, triple

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & \overset{*}{1} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore \hat{e}_{-1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; \text{ only one.}$$

6.1.2 Show that the eigenvalues of a triangular matrix are the diagonal element of the matrix.

Recall that if $T = \begin{pmatrix} t_{11} & & \\ & \ddots & \\ 0 & & t_{nn} \end{pmatrix}$ is triangular, then

$\det T = t_{11} t_{22} \cdots t_{nn} \equiv \prod_{k=1}^{n} t_{kk}$, the product of the diagonals

Then
$$\det(T - \lambda I) = \begin{vmatrix} t_{11}-\lambda & & & \\ & t_{22}-\lambda & & \\ & & \ddots & \\ 0 & & & t_{nn}-\lambda \end{vmatrix} = \prod_{k=1}^{n} (t_{kk} - \lambda) = 0$$

$$\Rightarrow \boxed{\lambda_k = t_{kk} ; \; k = 1, \ldots, n}$$

6.1.10 Show that the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ will have complex evals. if $\theta \neq n\pi$.

$$\det(A - \lambda I) = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta =$$
$$= \cos^2\theta + \sin^2\theta - 2\lambda\cos\theta + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

which is complex unless $\theta = 2n\pi$ $(\lambda = 1)$ or $\theta = (2n+1)\pi$ $(\lambda = -1)$. The matrix A represents a rotation by angle θ, so it can only have real eigenvalues/vectors for $\theta = \pi + 2n\pi$ (evalue -1, reversal) or $\theta = 2n\pi$ (evalue 1, identity).

($e^{i\theta}$ rotates complex numbers by angle θ in $\mathbb{C}$-plane)

6.1.13 Let A be a 2x2 matrix and let $p(\lambda) = \lambda^2 + b\lambda + c$ be the characteristic polynomial of A. Show that $b = -\mathrm{tr}(A)$ and $c = \det(A)$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$$

$$\Rightarrow \lambda^2 - (a_1 + a_2)\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\boxed{\lambda^2 - (\mathrm{Tr}\,A)\lambda + \det A = 0}$$

Solutions
Math. 314 (p. 323)
6.2 (1c, 2c)

6.2.1c

$$\left.\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}\right\} \therefore \frac{d}{dt}\underline{y} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\underline{y} = A\underline{y}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda - 5) = 0$$

$$\lambda_1 = 0: \quad \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 - 2x_2 = 0: \ x_1 = 2, x_2 = 1 \ : \ \underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 5: \quad \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow -4x_1 - 2x_2 = 0: \ x_1 = 1, x_2 = -2 : \ \underline{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\underline{y}(t) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

6.2.2c

$$\begin{pmatrix} y_1(0) = y_2(0) = \\ = y_3(0) = 2 \end{pmatrix} \quad \left.\begin{aligned} y_1' &= 2y_1 - 6y_3 \\ y_2' &= y_1 - 3y_3 \\ y_3' &= y_2 - 2y_3 \end{aligned}\right\} \quad \underline{y}' = \begin{pmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}\underline{y} = A\underline{y}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} - 6\begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix}$$

$$= (2-\lambda)\lambda(2+\lambda) + 3(2-\lambda) - 6 = -\lambda^3 + \lambda = 0$$

$$\Rightarrow \lambda(1 - \lambda^2) = 0: \ \lambda = 0, +1, -1$$

$$\lambda_1 = 0: \begin{pmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{0} \Rightarrow \left.\begin{aligned} x_1 - 3x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \\ x_3 &= 1 \end{aligned}\right\} \quad \underline{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1: \begin{pmatrix} 1 & 0 & -6 \\ 1 & -1 & -3 \\ 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left.\begin{aligned} x_1 - 6x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} x_1 &= 6 \\ x_2 &= 3 \\ x_3 &= 1 \end{aligned}\right\} \quad \underline{v}_2 = \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -1: \begin{pmatrix} 3 & 0 & -6 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \left.\begin{aligned} 3x_1 - 6x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 1 \end{aligned}\right\} \quad \underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{y} = c_1\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\underline{y}(0) = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \rightarrow \left(\begin{array}{ccc|c} 3 & 6 & 2 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right) \rightarrow \begin{array}{c} \\ -2 \\ -3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 3 & 6 & 2 & 2 \end{array}\right) \rightarrow \begin{array}{c} \\ \\ -3 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -1 & -4 \end{array}\right)$$

use third row

$$\rightarrow \begin{array}{c} \\ \\ 1/2 \end{array}\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)\begin{array}{c} -1 \\ 1 \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)\begin{array}{c} -1 \\ \\ \\ \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow c_1 = 2, \ c_2 = -1, \ c_3 = 1: \quad \underline{y}(t) = 2\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ 1 \end{pmatrix}e^{t} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}e^{-t}$$