Set 27
Solutions
Math. 314
Fall '99

p.316 (6.3 — 1(a,d), 2(a,d), 27(a,c))

(1) Factor $A = XDX^{-1}$

(a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\lambda = 1$; $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\lambda = -1$; $\underline{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

these are orthogonal, so normalize: $\underline{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\underline{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Then $X = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$; $X^{-1} = X^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$;

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

(d) $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$; $\det(A-\lambda I) = (2-\lambda)(1-\lambda)(-1-\lambda) = 0$.

$\lambda = 2$: $\begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 5/2 \\ 0 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{u}_1$

$\lambda = 1$: $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$: $\underline{u}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; $\underline{u}_3 = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) \xrightarrow{-1/3} \left(\begin{matrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{matrix}\right|\left.\begin{matrix} 1 & 0 & -1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right) \xrightarrow{2} \left(\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{matrix}\right|\left.\begin{matrix} 1 & 2 & 5/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right)$

$\to \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right|\left.\begin{matrix} 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & 1/3 \end{matrix}\right)$

$$\Rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & 1/3 \end{pmatrix}$$

(2a) $A^6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^6 = X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^6 X^{-1} = X \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X^{-1}$

$= I$

(2d) $A^6 = X \begin{pmatrix} 2 & & \\ & 1 & \\ & & -1 \end{pmatrix}^6 X^{-1} = X \begin{pmatrix} 64 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X^{-1}$

$$= \begin{pmatrix} 64 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 5/3 \\ 0 & -1 & -1 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 64 & 126 & 105 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(24a) $A = \begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix}$; $\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & -1 \\ 6 & 3-\lambda \end{pmatrix} = (-2-\lambda)(3-\lambda) + 6 = 0$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0$

$\lambda = 1$: $\begin{pmatrix} -3 & -1 \\ 6 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1/3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda = 0$: $\begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \underline{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix} \Rightarrow X^{-1} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \end{pmatrix}$; $A = X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1}$

$A^n = X \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X^{-1} = A$. Then $e^A = I + A\left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots\right)$

$= I + A(e - 1)$

(24c) $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ $\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ -1 & -1-\lambda & -1 \\ 1 & 1 & 1-\lambda \end{pmatrix} \overset{!}{=} 0 \Rightarrow (1-\lambda)\big((-1-\lambda)(1-\lambda) + 1\big)$

$- \big((-1)(1-\lambda) + 1\big) + \big(-1 + (1+\lambda)\big) =$

$= (1-\lambda)(\lambda^2 - 1 + 1) - (\lambda - 1 + 1) + (-1 + 1 + \lambda) = \lambda^2(1-\lambda) = 0$

$\lambda = 1$: $\begin{pmatrix} 0 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ -1 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

So:

$$\lambda = 1: \quad \underline{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda = 0: \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

i.e. $\underline{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$X^{-1}: \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$\text{So} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{pmatrix} = X D X^{-1}$$

$$A^n = X D^n X^{-1} = X \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} X^{-1} = A$$

$$e^A = I + A + \frac{A^2}{2!} + \dots + \frac{A^n}{n!} + \dots$$

$$= I + A\left(1 + \frac{1}{2!} + \dots + \frac{1}{n!}\right) = I + A(e-1)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (e-1) \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} e & e-1 & e-1 \\ 1-e & 2-e & 1-e \\ e-1 & e-1 & e \end{pmatrix}$$

6.2.1d $\quad \underline{y}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A\underline{y}\ ; \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2 + 1 = 0$

$\Rightarrow \lambda = 1 \pm i$

$\lambda_{1,2}: \quad \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0} \Rightarrow (1-\lambda)x_1 - x_2 = 0 : \quad x_1 = 1, \quad x_2 = 1 - \lambda$

$\lambda_1 = 1 + i, \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}; \quad \lambda_2 = 1 - i, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

$$\underline{y} = c_1 \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+i)t} + c_2 \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(1-i)t}$$

Now $\quad \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^{(1+i)t} = e^t \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] (\cos t + i \sin t)$

$$= e^t \left[\begin{pmatrix} \cos t \\ \cos t \end{pmatrix} - \begin{pmatrix} 0 \\ -\sin t \end{pmatrix} \right] + i \left[\begin{pmatrix} \sin t \\ \sin t \end{pmatrix} + \begin{pmatrix} 0 \\ -\cos t \end{pmatrix} \right]$$

$$\underline{y}_1(t) = e^t \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} ; \quad \underline{y}_2(t) = e^t \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$$

$$\underline{y}(t) = c_1 e^t \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}$$