Math. 514
Fall 09
Set 28
solutions

p. 328, Sec. 6.4
2, 4(ace), 5(bcd)

(2) $z_1 = \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix}$, $z_2 = \begin{pmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

(a) $\|z_1\|^2 = \left(\frac{1-i}{2}\right)\left(\frac{1+i}{2}\right) + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right)$

$z_1^H z_1 = \left(\frac{1+1}{4}\right) + \left(\frac{1+1}{4}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$\|z_2\|^2 = \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} = 1$

$\| = z_2^H z_2$

$z_1^H z_2 = \left(\frac{1-i}{2}\right)\left(\frac{i}{\sqrt{2}}\right) + \left(\frac{1+i}{2}\right)\left(-\frac{1}{\sqrt{2}}\right) = \frac{i+1}{2\sqrt{2}} + \frac{-1-i}{2\sqrt{2}} = 0$

(b) $(z_1\ z_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = z \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Z^H z$

$\begin{pmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{2}} \\ \frac{1-i}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2+4i \\ -2i \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1+2i)(1-i) - i(1+i) \\ \frac{-i(2+4i) + 2i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1+2+i-i+1 \\ \frac{-2i+4+2i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{4}{\sqrt{2}} \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1/\sqrt{2} \end{pmatrix}$

(4a) $\begin{pmatrix} 1-i & 3 \\ 2 & 3 \end{pmatrix}$ not Hermitian (complex diag.)

$A^H A = \begin{pmatrix} 1+i & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1-i & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2+4 & 3+3i+6 \\ 3-3i+6 & 9+9 \end{pmatrix} = \begin{pmatrix} 6 & 9+3i \\ 9-3i & 18 \end{pmatrix}$

$A A^H = \begin{pmatrix} 1-i & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1+i & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 2+9 & - \\ - & - \end{pmatrix}$

$A^H A \neq A A^H$: not normal

(4c) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ not Hermitian; but it is orthogonal, so $A^H A = I = A A^H$

so it is <u>normal</u>.

(4e) $A = \begin{pmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{pmatrix}$; $A^H = \begin{pmatrix} 0 & -i & -1 \\ -i & 0 & 2-i \\ 1 & -2-i & 0 \end{pmatrix} = -A$

A is skew-Hermitian;

so $A A^H = -A^2 = A^H A \Rightarrow$ A normal.

(5b) $\begin{pmatrix} 1 & 3+i \\ 3-i & 4 \end{pmatrix}$: $\det \begin{vmatrix} 1-\lambda & 3+i \\ 3-i & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (3+1)$

Hermitian $= \lambda^2 - 5\lambda = (\lambda - 5)\lambda = 0$: $\lambda = 5, 0$

$\lambda = 5$: $\begin{pmatrix} 1-5 & 3+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3+i \\ 4 \end{pmatrix} \rightarrow \frac{1}{\sqrt{26}} \begin{pmatrix} 3+i \\ 4 \end{pmatrix}$

$\|\underline{u}_1\|^2 = (3-i)(3+i) + 4^2 = (9+1) + 16 = 26$

$\lambda = 0$: $\begin{pmatrix} 1 & 3+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3+i \\ -1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{11}} \begin{pmatrix} 3+i \\ -1 \end{pmatrix}$

$\|u_2\|^2 = (3-i)(3+1) + 1^2 = (9+1) + 1 = 11$

i.e. $X = \begin{pmatrix} \frac{3+i}{\sqrt{26}} & \frac{3+i}{\sqrt{11}} \\ \frac{4}{\sqrt{26}} & \frac{-1}{\sqrt{11}} \end{pmatrix}$, $X^{-1} = \begin{pmatrix} \frac{3-i}{\sqrt{26}} & \frac{4}{\sqrt{26}} \\ \frac{3-i}{\sqrt{11}} & \frac{-1}{\sqrt{11}} \end{pmatrix}$

$\parallel$

X^H

6.4.5 c

$$\begin{vmatrix} 2-\lambda & i & 0 \\ -i & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left((2-\lambda)^2 + i^2\right)$$

$$= (2-\lambda)\left((2-\lambda)^2 - 1\right)$$

$$= (2-\lambda)(\lambda-3)(\lambda-1)$$

$\lambda = 1, 2, 3$

$\lambda = 1$: $\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $c = 0$, $a = i$, $b = -1$ $\rightarrow \begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $c = 1$, $a = b = 0$: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -1 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $c = 0$, $a = i$, $b = 1$: $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$

i.e. $\vec{V} = \begin{pmatrix} i & 0 & i \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \vec{u} = \begin{pmatrix} i/\sqrt{2} & 0 & i/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$

$D = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$

6.4.5 d
Continued

$$U = \begin{pmatrix} -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}, \quad D = \begin{pmatrix} 0 & & \\ & 3 & \\ & & 5 \end{pmatrix}$$

$$\boxed{A = UDU^H}$$

6.4.5 d) $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (2-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -2 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3-\lambda & -2 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left((3-\lambda)^2-4\right) - (3-\lambda) - 2 - 2 - (3-\lambda) = 0$$

$$= (2-\lambda)\left(\lambda^2-6\lambda+5\right) + 2\lambda - 10 =$$

$$-\lambda^3 + 8\lambda^2 - 15\lambda = 0 \Rightarrow -\lambda\left(\lambda^2 - 8\lambda + 15\right) = 0$$

$$(\lambda-3)(\lambda-5)$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-5) = 0 \quad : \quad \lambda = 0, 3, 5$$

$$\lambda = 0: \begin{array}{c} \\ -\tfrac{1}{2} \\ -\tfrac{1}{2} \end{array}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{v}_0 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \underline{u}_0 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 3: \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\underline{v}_5 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \underline{u}_5 = \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\lambda = 5: \begin{array}{c} \\ \tfrac{1}{3} \\ \tfrac{1}{3} \end{array}\begin{pmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 1 & 1 \\ 0 & -5/3 & -5/3 \\ 0 & -5/3 & -5/3 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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