Math. 314
Fall 09
Set ~~IV~~ V

p. 69, Sec. 1.4
4(abc), 8(ac)

p.65. 1.4.4(abc) $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}; \begin{pmatrix}1&-3\\0&1\end{pmatrix}; \begin{pmatrix}1/2&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Find E so that $AE = B$

(a) $A = \begin{pmatrix}4&1&3\\2&1&4\\1&3&2\end{pmatrix}$; $B = \begin{pmatrix}3&1&4\\4&1&2\\2&3&1\end{pmatrix}$; $E \equiv P_{13} = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$

B has the same columns as A, with columns 1 and 3 interchanged. This is accomplished by multiplying A on the ~~left~~ right by the matrix P_{13}, which is the matrix that permutes columns $1 \leftrightarrow 3$.

$$\begin{pmatrix}4&1&3\\2&1&4\\1&3&2\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} = \begin{pmatrix}0\cdot4+0\cdot1+1\cdot3 & \cdot & \cdot\\0\cdot2+0\cdot1+1\cdot4 & \cdot & \cdot\\0\cdot1+0\cdot3+1\cdot2 & \underbrace{\cdot}_{\text{done similarly}} & \cdot\end{pmatrix} = \begin{pmatrix}3&\cdot&\cdot\\4&\cdot&\cdot\\2&\cdot&\cdot\end{pmatrix}$$

(b) $A = \begin{pmatrix}2&4\\1&6\end{pmatrix}$; $B = \begin{pmatrix}2&-2\\1&3\end{pmatrix}$

The second column of B is constructed by the columns of A: $-3\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$

So: $A\begin{pmatrix}1&-3\\0&1\end{pmatrix} = B$

To do this without guessing, back to systems.

We need to solve: $A\underline{x}_1 = \begin{pmatrix}2\\1\end{pmatrix}$; $A\underline{x}_2 = \begin{pmatrix}-2\\3\end{pmatrix}$;

Write as one: $A(\underline{x}_1, \underline{x}_2) = B$ or

$$\begin{pmatrix}2&4\\1&6\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}\\x_{21}&x_{22}\end{pmatrix} = \begin{pmatrix}2&-2\\1&3\end{pmatrix}$$

(i.e. solve the same matrix with two different r.h.s.)

$$\begin{matrix}\\-1/2\end{matrix}\left(\begin{array}{cc|cc}2&4&2&-2\\1&6&1&3\end{array}\right) \to \left(\begin{array}{cc|cc}2&4&2&-2\\0&4&0&4\end{array}\right) \to \left(\begin{array}{cc|cc}1&2&1&-1\\0&1&0&1\end{array}\right)^{-2}$$

$$\to \left(\begin{array}{cc|cc}1&0&1&-3\\0&1&0&1\end{array}\right) \quad \text{i.e.} \quad \underline{X} = \begin{pmatrix}1&-3\\0&1\end{pmatrix} : \quad \boxed{A\underline{X} = B}$$

(c) $A = \begin{pmatrix} 4 & -2 & 5 \\ -2 & 4 & 2 \\ 6 & 1 & -2 \end{pmatrix}$; $B = \begin{pmatrix} 2 & -2 & 5 \\ -1 & 4 & 2 \\ 3 & 1 & -2 \end{pmatrix}$

Notice that the first column of B is $\frac{1}{2}$ * the first column of A, while the other two columns of B are equal the corresponding columns of A.
We multiply A on the right by the diagonal matrix $E = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

70, 1.48 a,c > Compute LU factorization

$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$

(a) $A = \begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} = U$

i.e. $\begin{pmatrix} 3 & 1 \\ 9 & 5 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}}_{= L}\underbrace{\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}}_{= U}$

(b) $A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ -3 & 1 & \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ -2 & 2 & 7 \end{pmatrix} \to$

$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ -2 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix} = U$

Then $\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}}_{E_3}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}}_{E_2}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1}\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$

So: $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 6 \\ -2 & 2 & 7 \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1^{-1}}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}_{E_2^{-1}}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}_{E_3^{-1}}}_{=} \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}}_{U}$

$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}}_{= L}\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}}_{= U}$

Ma 314
Spring '07
Set VI

1.4 (6ab, 12b)
A (4), B (6,7)
1.5 (4c*d*),
5 (abc*d*)
11 (a*, b*)

VI (1/3)

1.4.6 Given $A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$

(a) Find elem. mat. E_1, E_2, E_3 such that $E_3 E_2 E_1 A = U$ with U upper triangular

(b) Determine inverses of E_i, $i=1,2,3$, and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is L? Verify $A = LU$.

(a) $\begin{array}{c} \\ 3 \\ 2 \end{array}\begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix} \to \begin{array}{c} \\ \\ -1 \end{array}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = U$

That is $E_1 = \begin{pmatrix} 1 & & \\ -3 & 1 & \\ -2 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1 & 1 \end{pmatrix}$ (notice I combined the first two into one matrix)

Check: $E_1 A = \begin{pmatrix} 1 & & \\ -3 & 1 & \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix}$

$E_2(E_1 A) = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & +1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = U$

(b) Now $L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 2 & -1 & 1 \end{pmatrix}$

change signs of off-diagonal terms to get inverses

Check: $LU = \begin{pmatrix} 1 & & \\ 3 & 1 & \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}$

1.4.12b Given $A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}$

(b) $XA + B = C \Rightarrow XA = C - B \Rightarrow X = (C-B)A^{-1}$,

$\cancel{A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}}$ $\cancel{\left[\begin{pmatrix} 5 & 3 & | & 1 & 0 \\ 3 & 2 & | & 0 & 1 \end{pmatrix}\right.}$ [illegible]

$\to$

$$A^{-1}: \tfrac{3}{5}\left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right) \to \begin{array}{c}1/5\\5\end{array}\left(\begin{array}{cc|cc} 5 & 3 & 1 & 0 \\ 0 & 1/5 & -3/5 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 3/5 & 1/5 & 0 \\ 0 & 1 & -3 & 5 \end{array}\right) \xrightarrow{3/5} \left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 5 \end{array}\right)$$

$$\therefore \; A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \; ; \quad C - B = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix} - \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -8 & -1 \end{pmatrix}$$

Then $X = \begin{pmatrix} -2 & -4 \\ -8 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 8 & -14 \\ -13 & 19 \end{pmatrix}$

A.4 If A, B nonsingular $n\times n$ then $A+B$ also nonsingular:

False, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A+B = 0$

Even if it is true, $(A+B)^{-1} \neq A^{-1} + B^{-1}$

Say $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

$A+B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $(A+B)^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix} \neq A^{-1}+B^{-1} = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}$

B.6 Let A 3×3, $\vec{b} = 3\vec{a}_1 + \vec{a}_2 + 4\vec{a}_3$

Will the system be consistent?

YES, since $A\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \vec{b}$, we already know it is solvable.

B.7 Let A 3×3, suppose $\vec{a}_1 - 3\vec{a}_2 + 2\vec{a}_3 = 0$

Is A nonsingular?

No, since $A\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ i.e. the homogeneous system has a nontrivial solution

(If A nonsingular, then $A\hat{x} = 0$ has only the trivial solution $\vec{x} = 0$)

1.5.4 (c*, d*) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\quad D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},\quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{array}\right),$$

(c) $$\begin{pmatrix} C & O \\ O & C \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} CB_{11} & CB_{12} \\ CB_{21} & CB_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 3 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

C ~~adds~~ subtracts the 1st row from 2nd

(d) $$\begin{pmatrix} E & O \\ O & E \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} EB_{11} & EB_{12} \\ EB_{21} & EB_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 3 & 2 & 1 & 2 \\ 3 & 1 & 1 & 1 \end{array}\right)$$

E exchanges 1st & 2nd row

1.5.5 (c*d*) Perform each of the following block multiplications

(c) $$\left(\begin{array}{cc|cc} \frac{3}{5} & -\frac{4}{5} & 0 & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right)\left(\begin{array}{cc|c} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} A_{11}B_{11} & A_{11}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \hline 0 \quad 0 & A_{22}B_{22} \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = I^{3\times 3}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \qquad A_{11}B_{11} = \begin{pmatrix} \frac{9+16}{25} & \frac{12-12}{25} \\ \frac{12-12}{25} & \frac{9+16}{25} \end{pmatrix} = I$$

(d) $$\left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)\left(\begin{array}{cc} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ \hline 4 & -4 \\ 5 & -5 \end{array}\right) = \left(\begin{array}{c} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -4 \\ 5 & -5 \end{pmatrix} \end{array}\right) = \left(\begin{array}{cc} 3 & -3 \\ 2 & -2 \\ 1 & -1 \\ \hline 5 & -5 \\ 4 & -4 \end{array}\right)$$

(1.5.11) Let $A = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}$, where A_{ij} is ~~4x4~~ $n\times n$ metrix

If A_{11}, A_{22} are nonsingular, consider $B = \begin{pmatrix} A_{11}^{-1} & C \\ O & A_{22}^{-1} \end{pmatrix}$

Then $$AB = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}\begin{pmatrix} A_{11}^{-1} & C \\ O & A_{22}^{-1} \end{pmatrix} = \begin{pmatrix} A_{11}A_{11}^{-1} & A_{11}C + A_{12}A_{22}^{-1} \\ O & A_{22}A_{22}^{-1} \end{pmatrix}$$

since $A_{11}A_{11}^{-1} = A_{22}A_{22}^{-1} = I^{4\times 4}$, then $B = A^{-1}$ if

$A_{11}C + A_{12}A_{22}^{-1} = O \Rightarrow C = -A_{11}^{-1}A_{12}A_{22}^{-1}$. For this choice of C we have $BA = I^{8\times 8}$ as well, so $B = A^{-1}$, A nonsingular.