Math. 314 Fall 09 Set VII

A, 2.2(2, 3(bdf), 4)

p. 102. 2.2.2

(a) $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix}$;

The determinant of A equals the product of the pivots. We reduce to triangular form: (sign changes each time we permute rows)

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{pmatrix} \xrightarrow{P} \begin{array}{r} \\ -1 \\ 2 \\ 0 \end{array}\begin{pmatrix} 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\rightarrow \begin{array}{r} \\ \\ 2 \\ 1 \end{array}\begin{pmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 2 & -1 & -3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{array}{r} \\ \\ \\ -1 \end{array}\begin{pmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -2 & -3 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\det A = (-1)^{\#\text{permutations}} \cdot 1 \cdot (-1) \cdot 5 \cdot 2 = 10$$

(b) $$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix} \begin{array}{l} \\ \\ (1\ 1\ 1\ 1) + (-2\ -2\ 3\ 3) \\ (1\ 1\ 1\ 1) + (1\ 2\ -2\ -3) \end{array}$$

$= (-1)^2 \det A$ (second row permuted with third, then with fourth: two sign changes)

$= \det A$ (second row added to third & fourth rows; no change in value)

$$= 2 \det A = 20$$

3)(bdf) Compute determinant, state whether matrix is nonsingular.

b) • $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$; $\det A = 3 \cdot 2 - 4 \cdot 1 = 6 - 4 = 2 \neq 0$ nonsing

$$\begin{array}{r} \\ -2 \\ -1 \end{array}\begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$; $\det A = 2 \cdot 1 \cdot 1 = 2 \neq 0$ nonsingular

$\longrightarrow$

f) $A = \begin{array}{c} \\ -2 \\ \\ \end{array}\begin{pmatrix} 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix} \rightarrow \begin{array}{c} \\ 1/3 \\ \\ \end{array}\begin{pmatrix} 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix} \rightarrow \begin{array}{c} \\ \\ -3 \end{array}\begin{pmatrix} 0 & -3 & 1 & 0 \\ 0 & 0 & 7/3 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix} \rightarrow$

$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 7/3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; $\det A = 1 \cdot (-3) \cdot 7/3 \cdot 0 = 0$

<u>singular</u>

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p. 97, 2.2.4 Find all values of c for which

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{pmatrix}$ is singular

$A \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 8 & c-1 \\ 0 & c-1 & 2 \end{pmatrix}$; $\det A = 1 \cdot \det \begin{pmatrix} 8 & c-1 \\ c-1 & 2 \end{pmatrix}$

$\Rightarrow \det A = 16 - (c-1)^2$

For A to be singular, we need $\det A = 0 \Rightarrow$

$(c-1)^2 = 16 \Rightarrow c-1 = \pm 4 \Rightarrow c = 1 \pm 4 = \begin{cases} 5 \\ -3 \end{cases}$

<u>Determinants</u>

(1) Definition

Math. 314, Spring 07
Set VIII
Sec 2.3 (2(a,e), 3, 14)
p. 109

2.3.2 Use Cramer's rule to solve the systems

(a) $x_1 + 2x_2 = 3$, $3x_1 - x_2 = 1$ $\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\operatorname{Adj} A = \begin{pmatrix} -1 & -2 \\ -3 & 1 \end{pmatrix}$

$\det A = -7$

$M_{11} = -1,\ M_{12} = -3,\ M_{21} = -2,\ M_{22} = 1$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{\det A}\operatorname{Adj} A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 5 \\ 8 \end{pmatrix} =$$

(e)
$$\begin{aligned} x_1 + x_2 \qquad\qquad &= 0 \\ x_2 + x_3 - 2x_4 &= 1 \\ x_1 \qquad + 2x_3 + x_4 &= 0 \\ x_1 + x_2 \qquad + x_4 &= 0 \end{aligned}$$

$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$; $A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix}$

$X = A^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \\ A_{24} \end{pmatrix}$ we don't need to find other entries.

$\det M_{21} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$, $\det M_{22} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$, $\det M_{23} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1$, $\det M_{24} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{vmatrix} = 0$

$A_{21} = (-1)^{2+1}2 = -2$, $A_{22} = 2$, $A_{23} = (-1)^{2+3}(-1) = 1$, $A_{24} = 0$

$\det A = a_{21}A_{21} + \dots + a_{24}A_{24} = 0\cdot(-2) + 1\cdot 2 + 1\cdot(+1) + (-2)\cdot 0 = 3$

$(x_1, x_2, x_3, x_4) = \frac{1}{3}(-2, 2, 1, 0)$

2.3.4. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$; $\cancel{(A^{-1})_{23} = \frac{1}{\det A}(\operatorname{Adj} A)_{23} = \frac{(-1)^{3+2}\det M_{32}}{\det A}}$

$M_{32} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$; $\det M_{32} = -2$; $M_{31} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$; $\det M_{31} = -1$,

$M_{33} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\det M_{33} = -1$; $\det A = (-1)^{3+1}3\cdot(-1) + (-1)^{3+2}(4)(-2) + (-1)^{3+3}5(-1)$
$= -3 + 8 - 5 = 0$

$\det A = 0$

2.3.3 $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{pmatrix}$; find $(A^{-1})_{2,3}$

$$(A^{-1})_{2,3} = \frac{1}{\det A}(\operatorname{Adj} A)_{2,3} = \frac{(-1)^{2+3}\det M_{32}}{\det A}$$

$$\det M_{31} = \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 2, \quad \det M_{32} = \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 3, \quad \det M_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

$$\det A = (-1)^{3+1}\underbrace{(1)(2)}_{a_{31}\,|M_{31}|} + (-1)^{3+2}\underbrace{(2)(3)}_{a_{32}\,|M_{32}|} + (-1)^{3+3}\underbrace{(2)\cdot(4)}_{a_{33}\,|M_{33}|} = 2 - 6 + 8 = 4$$

$$(A^{-1})_{2,3} = -\frac{3}{4}$$

(14) In mssge.: 0 = space, 1 = A, 2 = B, .., 26 = Z.
Decode [−19, 19, 25, −21, 0, 18, −18, 15, 3, 10, −8, 3, −2, 20, −7, 12]

$$A = \begin{array}{c} \\ -1 \\ 0 \\ -1 \end{array} \left[\begin{array}{cccc|cccc} -1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} -1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{array}{c} \\ \\ \\ -1 \end{array} \left[\begin{array}{cccc|cccc} -1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \to$$

$$\to \begin{array}{c} -1* \\ -1* \\ -1* \\ \\ \end{array} \left[\begin{array}{cccc|cccc} -1 & -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 1 & -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \\ 1 \\ -1 \\ \\ \end{array} \to \left[\begin{array}{cccc|cccc} 1 & 1 & -2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} 2 \\ 2 \\ \\ \\ \end{array} \to$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \quad \text{i.e.} \quad A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$A^{-1}\,\text{MSSG} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -19 & 0 & 3 & -2 \\ 19 & 18 & 10 & 20 \\ 25 & -18 & -8 & -7 \\ -21 & 15 & 3 & 12 \end{pmatrix} = \begin{pmatrix} 4 & 15 & 8 & 23 \\ 15 & 21 & 15 & 15 \\ 0 & 18 & 13 & 18 \\ 25 & 0 & 5 & 11 \end{pmatrix}$$

4, 15, 0, 25, 15, 21, 18, 0, 8, 15, 13, 5, 23, 15, 18, 11
D O – Y O U R – H O M E W O R K