Math. 314
Spring '09
Set XII
Solutions

p. 191 (1, 16)
p. 137 (2, 4(a,d), 11(a))

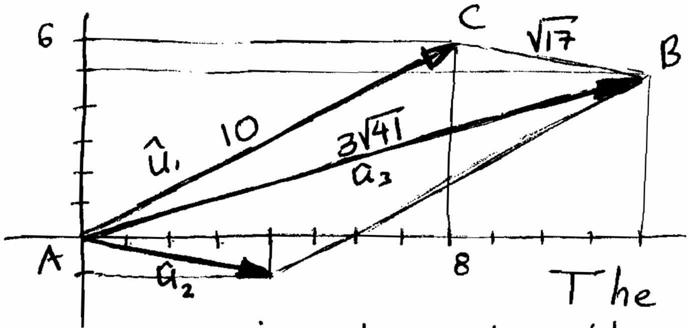
p.114, 3.1.1 $\hat{x}_1 = (8,6)^T,\ \hat{x}_2 = (4,-1)^T \in \mathbb{R}^2$

(a) $\|\hat{x}_1\| = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$

$\|\hat{x}_2\| = \sqrt{4^2+(-1)^2} = \sqrt{16+1} = \sqrt{17}$

(b) $\hat{x}_3 = \hat{x}_1 + \hat{x}_2 = (8,6)^T + (4,-1)^T = (8+4, 6-1)^T = (12,5)^T$

$\|\hat{x}_3\| = \sqrt{12^2+15^2} = \sqrt{144+225} = \sqrt{369} = 3\sqrt{41}$



$\hat{u}_3 = \hat{u}_1 + \hat{u}_2$

$\|\hat{u}_3\| \leqslant \|\hat{u}_1\| + \|\hat{u}_2\|$

$3\sqrt{41} < 10 \quad \sqrt{17}$

The length of a side of a triangle is not greater than the sum of the lengths of the other two sides (a straight segment between two points AB is the shortest of any other line joining A to B, such as ACB).

(p.116) 3.1.16 Each polynomial $p(x) = a_1 + a_2 x + \dots + a_n x^{n-1}$ in P_n can be associated to the vector $\hat{a} = (a_1, a_2, \dots, a_n)^T \in \mathbb{R}^n$. This association is called an <u>isomorphism</u> because, as vector spaces, P_n and $\mathbb{R}^n$ behave exactly alike under the above association.

(a) $\alpha p(x) = (\alpha a_1) + (\alpha a_2)x + \dots + (\alpha a_n)x^n \longleftrightarrow (\alpha a_1, \alpha a_2, \dots, \alpha a_n)^T =$

$\alpha (a_1, a_2, \dots, a_n)^T = \alpha \hat{a}$

(b) $p(x) + q(x) = (a_1 + a_2 x + \dots + a_n x^{n-1}) + (b_1 + b_2 x + \dots + b_n x^{n-1})$

$= (a_1 + b_1) + (a_2 + b_2)x + \dots + (a_n + b_n)x^{n-1}$

$\longleftrightarrow (a_1+b_1, a_2+b_2, \dots, a_n+b_n)^T =$

$= (a_1, a_2, \dots, a_n)^T + (b_1, b_2, \dots, b_n)^T = \hat{a} + \hat{b}$

p.123, 3.2.2 Determine whether or not the following are subspaces of $\mathbb{R}^3$:

(2a) $\{(x_1, x_2, x_3)^T \mid x_1 + x_3 = 1\}$: No, does not contain $\hat{0}$

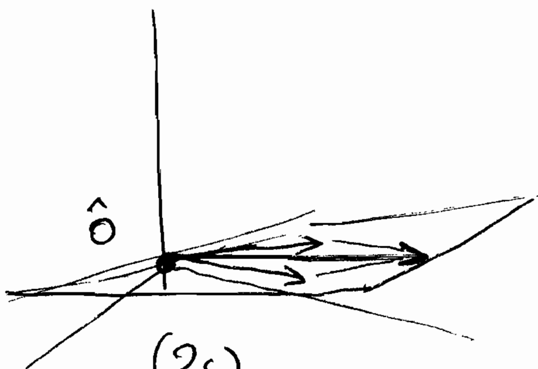
(b) $V = \{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\}$: Yes ; Contains $\hat{0}$

and if $\hat{x} \in \hat{V}$, $\hat{y} \in V \Rightarrow x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$

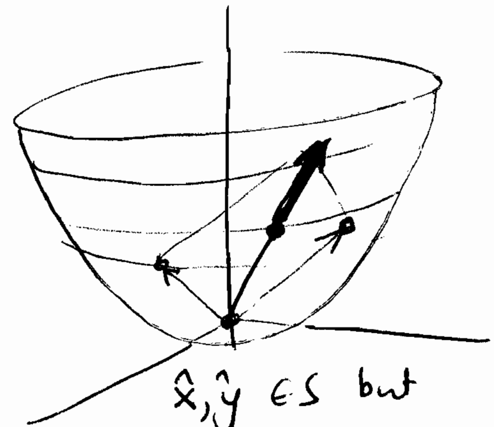
Then $\hat{x} + \hat{y} \in V$ since $x_1 + y_1 = x_2 + y_2 = x_3 + y_3$

(c) $V = \{(x_1, x_2, x_3)^T \mid x_3 = x_1 + x_2\}$ Yes : plane through the origin.

(d) $\{(x_1, x_2, x_3)^T \mid x_3 = x_1^2 + x_2^2\}$ No (parabolic bowl)



(2c)
Subspace
$(\alpha \hat{x} : \alpha x_3 = \alpha x_1 + \alpha x_2)$



$\hat{x}, \hat{y} \in S$ but
$\hat{x} + \hat{y} \notin S$

$(\alpha \hat{x} : \quad x_3 = x_1^2 + x_2^2$

Then $\quad \alpha x_3 \neq \alpha^2 x_1^2 + \alpha^2 x_2^2 \,)$

Not a subspace

p.123, 4(a,d) Determine the null space

(a) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \xrightarrow{-3/2} \begin{pmatrix} 2 & 1 \\ 0 & 1/2 \end{pmatrix}$ no null space (nonsingular)

$\longrightarrow$

p. 124, 3.2.4(d)

$$\begin{array}{c} \\ -2 \\ 1 \end{array}\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & -3 & 1 \\ -1 & -1 & 0 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -1 & -3 \end{pmatrix} \to \begin{array}{c} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccc} & * & & * \end{array} \end{array}$$

The nullspace of a matrix A, $\mathcal{N}(A)$, is the set of all vectors $\hat{n}$: $A\hat{n} = 0$ $\left(\mathcal{N}(A) = \{ \hat{n} \in \mathbb{R}^n \mid A\hat{n} = \hat{0}, A^{m \times n} \} \right)$.

There is one null vector for every free variable. We find it by setting each free variable $= 1$ in turn (while the others are set $= 0$). Here, the free variables are x_2, x_4 : so

~~(A) $\hat{n}_1$: set $x_2 = 1$, $x_4 = \emptyset$~~

Reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & -1 & -2 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{array}{c} \\ \cdot -1 \\ \\ \end{array} \to \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $x_1 = -x_2 - x_4$, $x_3 = -3x_4$ ~~N.B.~~

(i) Set $x_2 = 1$, $x_4 = 0$; then $x_1 = -1$, $x_3 = 0$: $\hat{n}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(ii) Set $x_2 = 0, x_4 = 1$; then $x_1 = -1$, $x_3 = -3$: $\hat{n}_2 = \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix}$

So : $\mathcal{N}(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}$

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p. 124, 3.2.11(a) $\hat{x}_1 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\hat{x}_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$, $\hat{x} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$, $\hat{y} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix}$

(a) To determine if $\hat{x} \in \text{span}\{\hat{x}_1, \hat{x}_2\}$ we must check if there are numpers α, β such that $\hat{x} = \alpha \hat{x}_1 + \beta \hat{x}_2$. But this is equivalent to the system $\begin{pmatrix} \hat{x}_1 & \hat{x}_2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \hat{x}$ or $\begin{pmatrix} -1 & 3 \\ 2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}$

or $\left.\begin{aligned} -\alpha + 3\beta &= 2 \\ 2\alpha + 4\beta &= 6 \\ 3\alpha + 2\beta &= 6 \end{aligned}\right\}$

over $\longrightarrow$

The augmented matrix is

$$\begin{array}{c} \\ \cdot 2 \\ 3 \end{array} \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 11 & 12 \end{array}\right) \quad \text{obviously inconsistent!}$$

So $\hat{x} \notin \operatorname{Span}(\hat{x}_1, \hat{x}_2)$.

Math. 314
Spring 09
Set VIII

p. 123 (4ad), 11(a), 14(a,c) / Sec. 3.2 — see set VII
p. 135, Sec. 3.3 2(bde), 4(ac)

14(a,c) Sec. 3.2, p. 123 Which of the following are spanning sets for P_3? Justify.

(a) $1 \to \begin{pmatrix}1\\0\\0\end{pmatrix}$, $x^2 \to \begin{pmatrix}0\\0\\1\end{pmatrix}$, $x^2-2 \to \begin{pmatrix}-2\\0\\1\end{pmatrix}$

To span P_3 we must be able to find α, β, γ such that

$$\alpha \cdot 1 + \beta \cdot x^2 + \gamma (x^2-2) = a_1 + a_2 x + a_3 x^2$$

But then $\begin{pmatrix}1 & 0 & -2\\0 & 0 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ for any a_1, a_2, a_3

and that is clearly not possible unless $a_2 \neq 0$. <u>NO</u>

(c) $\{x+2, x^2-1\}$: Now $\begin{pmatrix}2 & -1\\1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$

But now there can be at most two pivots, so that system does not have a solution for an arbitrary rhs:

$$\begin{matrix}\\-\tfrac12\\\\\end{matrix}\left(\begin{array}{cc|c}2 & -1 & a_1\\1 & 0 & a_2\\0 & 1 & a_3\end{array}\right) \to \begin{matrix}\\\\-2\end{matrix}\left(\begin{array}{cc|c}2 & -1 & a_1\\0 & \tfrac12 & a_2-\tfrac12 a_1\\0 & 1 & a_3\end{array}\right) \longrightarrow \left(\begin{array}{cc|c}2 & -1 & a_1\\0 & \tfrac12 & a_2-\tfrac12 a_1\\0 & 0 & a_3-\tfrac12 a_2+\tfrac14 a_1\end{array}\right)$$

and there can only be a solution if $a_3 - \frac12 a_2 + \frac14 a_1 = 0$,
i.e. the set does not span P_3

(p. 135) Sec. 3.3 2(bde) Determine whether or not the following vectors are linearly independent in $\mathbb{R}^3$.

(b) $\begin{pmatrix}1 & 0 & 1 & 1\\0 & 1 & 0 & 2\\0 & 1 & 1 & 3\end{pmatrix}$ Here we are guaranteed to find <u>one free variable at least</u> (since there can be no more than <u>three</u> pivots), so vectors are dependent.

(d) $\begin{matrix}\\-\tfrac12\\1\end{matrix}\begin{pmatrix}2 & -2 & 4\\1 & -1 & 2\\-2 & 2 & -4\end{pmatrix}$ Here $\det A = 0$ since the columns are multiples of the first, so vectors are dependent.

(col 2 = $-1 *$ col 1; col 3 = $2 *$ col 1 $\Rightarrow$ columns dependent: $\underline{a}_1 + \underline{a}_2 = \underline{0}$, $\underline{a}_3 - 2\underline{a}_1 = \underline{0}$, $\underline{a}_3 + 2\underline{a}_2 = \underline{0}$)

(e) $\begin{array}{c} \\ -1 \\ -3 \end{array}\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 3 & 1 \end{pmatrix} \longrightarrow \begin{array}{c} \\ \\ -\frac{1}{2} \end{array}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$ no free variables $\Rightarrow$ Columns independent

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p.135, sec.3.3 4(a,c)

Determine whether or not the following vectors are linearly independent in $R^{2\times2}$

(a) $\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{=\,\underline{u}_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=\,\underline{u}_2}$: $\alpha \underline{u}_1 + \beta \underline{u}_2 = \begin{pmatrix} \alpha & \beta \\ \alpha & \alpha \end{pmatrix} = \underline{0}$

$\Rightarrow \alpha = 0, \beta = 0$: <u>independent</u>

(no way to get a zero unless $\alpha = \beta = 0$)

(c) $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=\,\underline{u}_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=\,\underline{u}_2}, \underbrace{\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}}_{=\,\underline{u}_3}$: $\alpha \underline{u}_1 + \beta \underline{u}_2 + \gamma \underline{u}_3 = \begin{pmatrix} \alpha+2\gamma & \beta+3\gamma \\ 0 & \alpha+2\gamma \end{pmatrix} = \underline{0}$

$\Rightarrow \alpha + 2\gamma = 0, \ \beta + 3\gamma = 0$ system of two eqns. in α, β, γ.

Then $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \cancel{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}}$ <u>γ is free</u>

i.e. $\alpha = -2\gamma, \ \beta = -3\gamma$ gives a nontrivial solution.

i.e: $-2\gamma \cdot \underline{u}_1 - 3\gamma \underline{u}_2 + \gamma \underline{u}_3 = 0 \Rightarrow \boxed{-2\underline{u}_1 - 3\underline{u}_2 + \underline{u}_3 = 0}$

dependent
