314 '03-MIDTERM 2

Name:____

December 13, 2003

1 < $30pts$ >

In each of the following answer true if the statement is always true and false otherwise. Justify your answer: in the case of a true statement, explain or prove your answer; in the case of a false statement, give an example to show that the statement is not always true. No credit without justification!

a [F] If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in R^n and the vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is equal to the vector projection of $\mathbf{y}$ onto $\mathbf{x}$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.
(*) The vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is parallel to $\mathbf{y}$.
The vector projection of $\mathbf{y}$ onto $\mathbf{x}$ is parallel to $\mathbf{x}$.
If these two vector projections are equal (and different from zero), then $\mathbf{x}$ is parallel to $\mathbf{y}$ and the two are linearly dependent. If both projections are equal to the zero vector, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal, i.e. independent.

b [F] It is possible to find a nonzero vector $\mathbf{y}$ in the column space of A such that $A^T\mathbf{y} = \mathbf{0}$.
(*) if $A^T\mathbf{y} = \mathbf{0}$, then $y \in \mathcal{N}(A^T)$. But the left null space, $\mathcal{N}(A^T)$, is the orthogonal complement of the column space, $\mathcal{R}(A)$. The only vector that can belong to both simultaneously is the zero vector.

c [T] If $\mathcal{N}(A) = \{\mathbf{0}\}$, then the system $A\mathbf{x} = \mathbf{b}$ will have a unique least squares solution.
(*) Because if A has no null vectors, then neither will A^TA and the normal equations will have no free variables.

d [T] If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal set of vectors in R^n and $U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$, then $U^T U = I_k$ (the $k \times k$ identity matrix).
(*) Since the $\mathbf{u}_i$ are k orthonormal vectors in R^n, there cannot be more than n of them, i.e. $k \leq n$. When we multiply U on the left by U^T to form $U^T U$, the diagonal element $i - i$ is equal to $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = \mathbf{u}_i^T \mathbf{u}_i = 1$, $i = 1, 2, \ldots, k$ while all offdiagonal elements are zero, since they are of the form $\mathbf{u}_i^T \mathbf{u}_j$ with $i \neq j$.

e [F] If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal set of vectors in R^n and $U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k)$, then $UU^T = I_n$ (the $n \times n$ identity matrix).
(*) The rows of U are not an orthonormal set in general (unless $n = k$ in which case U becomes an orthogonal matrix), since there are n of them but they are vectors in R^k. They cannot even be independent if $n > k$.

f [T] If A is an $n \times n$ matrix, then A and A^2 have the same eigenvectors.
(*) $A\mathbf{v} = \lambda \mathbf{v} \rightarrow A\,(A\mathbf{v} = \lambda \mathbf{v}) \Rightarrow A^2 \mathbf{v} = \lambda A \mathbf{v} = \lambda^2 \mathbf{v}$

g [F] If A is an $n \times n$ matrix, then A and A^T have the same eigenvectors.
(*) $A\mathbf{v} = \lambda \mathbf{v} \rightarrow \mathbf{v}^T A^T = \lambda \mathbf{v}^T$. No reason why the left and right eigenvectors of a non-symmetric matrix A should be the same. (consider a 2x2 example with a nonsymmetric matrix A)

h [F] The rank of an $n \times n$ matrix A is equal to the number of nonzero eigenvalues of A, where eigenvalues are counted according to multiplicity.
(*) Consider the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has 2 zero eigenvalues, but rank 1.

i [T] If an $n \times n$ matrix A has Schur decomposition $A = UTU^H$, then the eigenvalues of A are $t_{11}, t_{22}, \ldots, t_{nn}$.
(*) Since U is unitary, $U^H = U^{-1}$ so that $A = UTU^{-1}$ and A is similar to T. Since T is triangular, its eigenvalues are its diagonal elements.

j [T] If U is a unitary $n \times n$ matrix and λ is an eigenvalue of U, then $|\lambda| = 1$.
(*) Let $U\mathbf{v} = \lambda \mathbf{v}$ then $||U\mathbf{v}||^2 = |\lambda|^2\, ||\mathbf{v}||^2 \rightarrow \mathbf{v}^H U^H U \mathbf{v} = |\lambda|^2\, \mathbf{v}^H \mathbf{v} \rightarrow$ $|\lambda|^2 = 1$ since $U^H U = I$.

2 $< 15pts >$

Find a basis of the row space, column space, null space and left null space of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

and give the dimensions of each.
Solution

$$\begin{array}{c} \\ -3 \\ -2 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \to \begin{array}{c} 0 \\ 0 \\ -1 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & -2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Putting pieces together, we have shown that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

So;

$$\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \quad , \quad \mathcal{R}(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{N}(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \quad , \quad \mathcal{N}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \right\}$$

3 $< 10pts >$

Find all least squares solutions of the system

$$\begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

Solution

First form the normal equations, $A^T A x = A^T b$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 26 \end{bmatrix}$$

Form augmented matrix and proceed as usual when there are free variables:

$$\begin{array}{c} \\ \\ -2 \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 6 & 14 & 26 & 26 \end{array}\right] \to \begin{array}{c} \\ \\ -1 \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 0 & 14 & 14 & 14 \end{array}\right] \to \begin{array}{c} 1/3 \\ 1/14 \\ \\ \end{array} \left[\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

4 $< 10pts >$

Find the least squares fit by a quadratic function ($y = a * x^2 + b * x + c$) to the data

x	-1	0	1	2
y	0	2	5	16

Solution
We have $a * x_i^2 + b * x_i + c = y_i \ , \ i = 1, 2, 3, 4$. Writing as a system:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 16 \end{bmatrix}$$

Multiplying by the transpose of the matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \\ 16 \end{bmatrix}$$

or

$$\begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 69 \\ 37 \\ 23 \end{bmatrix}$$

Since

$$\begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix}^{-1} = \frac{1}{80} \begin{bmatrix} 20 & -20 & -20 \\ -20 & 36 & 12 \\ -20 & 12 & 44 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 5 & -5 & -5 \\ -5 & 9 & 3 \\ -5 & 3 & 11 \end{bmatrix}$$

we find

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 5 & -5 & -5 \\ -5 & 9 & 3 \\ -5 & 3 & 11 \end{bmatrix} \begin{bmatrix} 69 \\ 37 \\ 23 \end{bmatrix} = \begin{bmatrix} 9/4 \\ 57/20 \\ 19/20 \end{bmatrix} = \begin{bmatrix} 2.25 \\ 2.85 \\ 0.95 \end{bmatrix}$$

5 $< 10pts >$

Consider the inner product space $C[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

Find a, b so that $u_1(x) = a$ and $u_2(x) = bx$ form an orthonormal set and find the best approximation for the function $f(x) = e^{-x}$ in the interval $[-1, 1]$, i.e. find c_1, c_2 so that $e^{-x} = c_1 * u_1(x) + c_2 * u_2(x)$ is best in the sense of least squares.

Solution

We simply have

$$\langle a, bx\rangle = ab\int_{-1}^{1} xdx = ab\left.\frac{x^2}{2}\right|_{-1}^{1} = ab * 0 = 0$$

while

$$\langle a, a\rangle = a^2\int_{-1}^{1} dx = 2a^2 = 1 \text{ if we set } a = 1/sqrt2$$

$$\langle bx, bx\rangle = b^2\int_{-1}^{1} x^2dx = b^2\left.\frac{x^3}{3}\right|_{-1}^{1} = b^2\frac{2}{3} = 1 \text{ if we set } b = \sqrt{3/2}$$

So, for the projection to u_1, u_2 we simply need

$$e^{-x} = c_1u_1(x) + c_2u_2(x)$$

with

$$c_1 = \left\langle 1/\sqrt{2}, e^{-x}\right\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} e^{-x}dx = \frac{e^1 - e^{-1}}{\sqrt{2}}$$

and

$$c_2 = \left\langle \sqrt{3/2}x, e^{-x}\right\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} e^{-x}xdx = -\left.\sqrt{\frac{3}{2}}(x+1)e^{-x}\right|_{-1}^{1} = -\sqrt{6}e^{-1}$$

6 $< 15pts >$

Let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 13 \\ 98 \end{bmatrix}$$

(a) (5pts) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.
(b) (5pts) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.
(c) (5pts) Solve the least squares problem

$$A\mathbf{x} = \mathbf{b} \ .$$

Solution
(a) We follow the GS-process on the vectors $\mathbf{a}_1, \mathbf{a}_2$, the columns of A:

$$\begin{aligned}
r_{11} &= ||\mathbf{a}_1|| = \sqrt{1 + (-1)^2 + 1} = \sqrt{3} \\
\mathbf{q}_1 &= \frac{1}{r_{11}} \mathbf{a}_1 \\
r_{12} &= \langle \mathbf{a}_2, \mathbf{q}_1 \rangle = (1 * 1 + 1 * (-1) + 1 * 1)/\sqrt{3} = 1/\sqrt{3} \\
\mathbf{q}_2' &= \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \\
r_{22} &= ||\mathbf{q}_2'|| = 2\sqrt{6}/3 \\
\mathbf{q}_2 &= \frac{1}{r_{22}} \mathbf{q}_2' = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}
\end{aligned}$$

(b) The factorization works as follows:

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{r_{11}} \mathbf{a}_1 \to \mathbf{a}_1 = \mathbf{q}_1 r_{11} \\
\mathbf{q}_2 &= \frac{1}{r_{22}} (\mathbf{a}_2 - r_{12}\mathbf{q}_1) \to \mathbf{a}_2 = \mathbf{q}_1 r_{12} + \mathbf{q}_2 r_{22}
\end{aligned}$$

so that

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = QR$$

or

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \frac{4}{\sqrt{6}} \end{bmatrix} = QR$$

(c) We now solve the least squares problem:

$$QR\mathbf{x} = \mathbf{b} \quad \rightarrow \quad R\mathbf{x} = Q^T\mathbf{b}$$

$$\begin{bmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \frac{4}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 \\ 13 \\ 98 \end{bmatrix} = \begin{bmatrix} \frac{88}{\sqrt{3}} \\ \frac{127}{\sqrt{6}} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \frac{4}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{88}{\sqrt{3}} \\ \frac{127}{\sqrt{6}} \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 88 \\ 127 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 88 \\ \frac{127}{4} \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{225}{12} \\ \frac{127}{4} \end{bmatrix} = \begin{bmatrix} 18.75 \\ 31.75 \end{bmatrix}$$

(c') Alternatively: solve using normal equations:

$$A^TAx = A^Tb \rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 13 \\ 98 \end{bmatrix}$$

i.e.

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 88 \\ 114 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 88 \\ 114 \end{bmatrix}$$

giving

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{150}{8} \\ \frac{254}{8} \end{bmatrix} = \begin{bmatrix} 18.75 \\ 31.75 \end{bmatrix}$$

7 $< 10pts >$

Solve the initial value problem:

$$\begin{aligned} y_1' &= -y_2 \ , \ y_1(0) = 1 \\ y_2' &= \quad y_1 \ , \ y_2(0) = 2 \end{aligned}$$

Solution

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \to \mathbf{y} = e^{At}\mathbf{y}_0 \ = Ve^{Dt}V^{-1}\mathbf{y}_0,$$

$$A = VDV^{-1} \ , \ V = [\mathbf{v}_1, \mathbf{v}_2] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \to \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \ = \ \lambda^2 + 1 = 0 \to \lambda = i, -i$$

We find the eigenvectors

1. $\lambda_1 = i$

$$-i \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} -i & -1 \\ 0 & 0 \end{bmatrix} \to \mathbf{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

2. $\lambda_2 = -i$

$$i \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \to \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix} \to \mathbf{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Then

$$V = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \ , \ V^{-1} = \frac{-i}{2} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \ , \ e^{Dt} = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{y}(t) &= \left(\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right) \left(\frac{-i}{2} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \quad \text{acceptable answer} \\ &= \frac{-i}{2} \begin{bmatrix} e^{it} & e^{-it} \\ -ie^{it} & ie^{-it} \end{bmatrix} \begin{bmatrix} -2+i \\ 2+i \end{bmatrix} = \frac{-i}{2} \begin{bmatrix} (-2+i)e^{it} + (2+i)e^{-it} \\ -i(-2+i)e^{it} + i(2+i)e^{-it} \end{bmatrix} \\ &= \begin{bmatrix} i(e^{it} - e^{-it}) + \dfrac{e^{it} + e^{-it}}{2} \\ (e^{it} + e^{-it}) + \dfrac{-i}{2}(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} -2\sin t + \cos t \\ 2\cos t + \sin t \end{bmatrix} \end{aligned}$$