314 '03-Section 6.2 solutions

December 6, 2003

1 $\quad < 1c >$

$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= -2y_1 + 4y_2 \end{aligned}$$

Solution

$$\frac{d\mathbf{y}'}{dt} = A\mathbf{y} := \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \rightarrow \mathbf{y} = e^{At}\mathbf{y}_0 \;\; = Ve^{Dt}V^{-1}\mathbf{y}_0,$$

$$A = VDV^{-1}\ ,\ V = [\mathbf{v}_1, \mathbf{v}_2] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{v}_i = \lambda_i \mathbf{v}_i \rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} &= (1-\lambda)(4-\lambda) - (-2)(-2) = 0 \\ \lambda(\lambda - 5) = 0 &\rightarrow \lambda = 0, 5 \end{aligned}$$

We find the eigenvectors

1. $\lambda_1 = 0$

$$\begin{array}{c} \\ 2 \end{array} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2. $\lambda_2 = 5$

$$\begin{array}{c} \\ -1/2 \end{array} \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Then

$$V = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \; V^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \; e^{Dt} = \begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix}$$

so that

$$\begin{aligned}
\mathbf{y}(t) &= \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{5t} \end{bmatrix}\frac{-1}{5}\begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{y}_0 \\
&= \begin{bmatrix} 2 & e^{5t} \\ 1 & 2e^{5t} \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & -2 \end{bmatrix}\mathbf{y}_0 \\
&= \frac{-1}{5}\begin{bmatrix} 4-e^{5t} & -2-2e^{5t} \\ 2-2e^{5t} & -1-4e^{5t} \end{bmatrix}\mathbf{y}_0
\end{aligned}$$

This method finds not only the general solution but the "fundamental" solution matrix as well; from it the solution to any initial value problem is found simply by multiplying times the initial condition vector. To write a general solution, we can simply set:

$$\mathbf{y}(t) = Ve^{Dt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & e^{5t} \\ 1 & 2e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

2 $< 2c >$

Solve the initial value problem:

$$\begin{aligned}
y_1' &= 2y_1 - 6y_3 \;, \; y_1(0) = 2 \\
y_2' &= y_1 - 3y_3 \;, \; y_2(0) = 2 \\
y_3' &= y_2 - 2y_3 \;, \; y_3(0) = 2
\end{aligned}$$

Solution

$$\frac{d\mathbf{y}'}{dt} = A\mathbf{y} := \begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \to \mathbf{y} = e^{At}\mathbf{y}(0) \;= Ve^{Dt}V^{-1}\mathbf{y}(0),$$

$$A = VDV^{-1} \;, \; V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \quad , \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$A\mathbf{v}_i=\lambda_i\mathbf{v}_i \to \begin{vmatrix} 2-\lambda & 0 & -6\\ 1 & -\lambda & -3 \\ 0 & 1 & -2-\lambda\end{vmatrix} \quad = \quad (2-\lambda)(-2-\lambda)(-\lambda)+(-6+3(2-\lambda))=0$$

$$\lambda\left(\lambda^2-4\right)+3\lambda=\lambda\left(\lambda^2-1\right)=0 \quad\to\quad \lambda=0,\pm1$$

We find the eigenvectors

1. $\lambda_1=-1$

$$\begin{array}{r} \\ -1/3 \\ 0 \end{array}\begin{bmatrix} 3 & 0 & -6 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{array}{r} \\ \\ -1 \end{array}\begin{bmatrix} 3 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v}_1=\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

2. $\lambda_2=0$

$$\begin{array}{r} \\ -1/2 \\ 0 \end{array}\begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix} \to \mathbf{v}_2=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

3. $\lambda_3=1$

$$\begin{array}{r} \\ -1 \\ 0 \end{array}\begin{bmatrix} 1 & 0 & -6 \\ 1 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{array}{r} \\ \\ 1 \end{array}\begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \mathbf{v}_3=\begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$$

Then

$$V=\begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \quad V^{-1}=\frac{-1}{2}\begin{bmatrix} -1 & 3 & -3 \\ 2 & -4 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad e^{Dt}=\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{t} \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{y}(t) &= \left(\begin{bmatrix} 2 & 3 & 6 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{t} \end{bmatrix}\right)\left(\frac{-1}{2}\begin{bmatrix} -1 & 3 & -3 \\ 2 & -4 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 2e^{-t} & 3 & 6e^{t} \\ e^{-t} & 2 & 3e^{t} \\ e^{-t} & 1 & e^{t} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2e^{-t}+6-6e^{t} \\ e^{-t}+4-3e^{t} \\ e^{-t}+2-e^{t} \end{bmatrix} \end{aligned}$$