314 '03-QUIZ 10

Name:____

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1 $< 13pts >$

Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 7 \\ 13 \\ 18 \end{bmatrix}$$

(a) (9pts) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.
(b) (8pts) Factor A into a product QR, where Q has an orthonormal set of column vectors and R is upper triangular.
(c) (8pts) Solve the least squares problem

$$A\mathbf{x} = \mathbf{b} \ .$$

Solution
(a) We follow the GS-process on the vectors $\mathbf{a}_1, \mathbf{a}_2$, the columns of A:

$$\begin{aligned}
r_{11} &= ||\mathbf{a}_1|| = \sqrt{1 + 2^2 + 1} = \sqrt{6} \\
\mathbf{q}_1 &= \frac{1}{r_{11}}\mathbf{a}_1 \\
r_{12} &= \langle \mathbf{a}_2, \mathbf{q}_1 \rangle = (1 * 1 + 2 * (-1) + (-1) * 2)/\sqrt{6} = -3/\sqrt{6} \\
\mathbf{q}_2' &= \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
r_{22} &= ||\mathbf{q}_2'|| = \sqrt{9/2} = 3/\sqrt{2} \\
\mathbf{q}_2 &= \frac{1}{r_{22}}\mathbf{q}_2' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

(b) The factorization works as follows:

$$\begin{aligned}
\mathbf{q}_1 &= \frac{1}{r_{11}}\mathbf{a}_1 \to \mathbf{a}_1 = \mathbf{q}_1 r_{11} \\
\mathbf{q}_2 &= \frac{1}{r_{22}}(\mathbf{a}_2 - r_{12}\mathbf{q}_1) \to \mathbf{a}_2 = \mathbf{q}_1 r_{12} + \mathbf{q}_2 r_{22}
\end{aligned}$$

so that

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = QR$$

or

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{0}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & -\frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} = QR$$

(c) We now solve the least squares problem:

$$QR\mathbf{x} = \mathbf{b} \quad \to \quad R\mathbf{x} = Q^T\mathbf{b}$$

$$\begin{bmatrix} \sqrt{6} & -\frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 \\ 13 \\ 18 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1*7+2*13+18*(-1)}{\sqrt{6}} \\ \frac{1*7+0*13+18*1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{15}{\sqrt{6}} \\ \frac{25}{\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{6} & -\frac{3}{\sqrt{6}} \\ 0 & \frac{3}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{15}{\sqrt{6}} \\ \frac{25}{\sqrt{2}} \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 25 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{15}{3} \\ \frac{25}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{20}{3} \\ \frac{25}{3} \end{bmatrix}$$