314 '03-QUIZ 9

Name:____

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1 $< 12pts >$

Given the matrix

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

show that the columns of A form an orthonormal set and solve the least squares problem $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = [1, 3, 2, 4]^T$.

Solution

First show that $A^T * A = I_2$:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

this establishes A is orthogonal; then the least squares problem has solution

$$\hat{\mathbf{x}} = A^T b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

2 $< 13pts >$

Consider the inner product space $C[0,1]$ with inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

Show that $u_1(x) = 1$ and $u_2(x) = \sqrt{3}(2x-1)$ form an orthonormal set and find the best approximation for the function $f(x) = e^{-x}$ in the interval $[0,1]$, i.e. find c_1, c_2 so that $e^{-x} = c_1 * u_1(x) + c_2 * u_2(x)$ is best in the sense of least squares.

Solution

We showed orthogonality of these in class. We simply have

$$\left\langle 1, \sqrt{3}(2x-1)\right\rangle = \int_0^1 \sqrt{3}(2x-1)dx = \sqrt{3}(x^2-x)\Big|_0^1 = \sqrt{3}(1-1) = 0$$

while

$$\langle 1, 1\rangle = \int_0^1 dx = 1$$

$$\left\langle \sqrt{3}(2x-1), \sqrt{3}(2x-1)\right\rangle = \int_0^1 3(2x-1)^2 dx = 3\int_0^1 (4x^2-4x+1)dx = 3\left(\frac{4}{3}-\frac{4}{2}+1\right) = 1$$

So, for the projection to u_1, u_2 we simply need

$$e^{-x} = c_1 u_1(x) + c_2 u_2(x)$$

with

$$c_1 = \left\langle 1, e^{-x}\right\rangle = \int_0^1 e^{-x}dx = 1 - e^{-1}$$

and

$$c_2 = \left\langle \sqrt{3}(2x-1), e^{-x}\right\rangle = \int_0^1 e^{-x}\sqrt{3}(2x-1)dx = -\sqrt{3}(2x+1)e^{-x}\Big|_0^1 = \sqrt{3}(1-3e^{-1})$$