Math. 316, ODEs
Exam I — Spring 2004

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NAME: ...

1. (20pts.) Find the inverse Laplace transforms

(a) (6pts)

$$F(s) = \frac{1}{(s+1)(s-1)(s+2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2} \to Ae^{-t} + Be^{t} + Ce^{-2t}$$

$$A = \frac{1}{(s-1)(s+2)}\Big|_{-1} = -\frac{1}{2}, \quad B = \frac{1}{(s+1)(s+2)}\Big|_{1} = \frac{1}{6}, \quad C = \frac{1}{s^2-1}\Big|_{-2} = \frac{1}{3}$$

(b) (7pts)

$$F(s) = \frac{s^2+1}{s^2(s^2-1)} \quad \text{let} \quad z = s^2 \quad = \frac{z+1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$

$$A = \frac{z+1}{z-1}\Big|_{0} = -1, \quad B = \frac{z+1}{z}\Big|_{1} = 2$$

$$= -\frac{1}{s^2} + \frac{2}{s^2-1}$$

$$\to -t + 2\sinh t$$

(c) (7pts)

$$F(s) = \frac{2s+7}{s^2+4s+13} = \frac{2s+7}{(s+2)^2+3^2} = \frac{2(s+2)+3}{(s+2)^2+3^2} \to 2e^{-2t}\cos 3t + e^{-2t}\sin 3t$$

$$s^2+4s+13 = (s^2+4s+4)+9$$

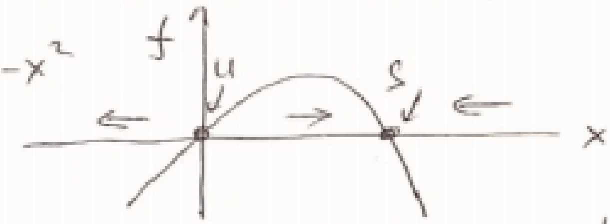
2. (20pts.) Solve the IVP

$$\frac{dx}{dt} = 4x - x^2 \ , \ x(0) = 1$$

and sketch the direction field and the solution. Identify the equilibrium points on the graph and state whether they are stable or unstable.

equlibrium pts: $\quad 4x - x^2 = x(4-x) = 0 \Rightarrow x = 0, 4$

$f(x) = 4x - x^2$



$$\int \frac{dx}{x(4-x)} = \int dt = t + c$$

$$\frac{1}{x(4-x)} = \frac{A}{x} + \frac{B}{4-x} = \frac{1}{4}\frac{1}{x} + \frac{1}{4}\frac{1}{4-x}$$

$$A = \frac{1}{4-x}\Big|_0 = \frac{1}{4}$$

$$B = \frac{1}{x}\Big|_4 = \frac{1}{4}$$

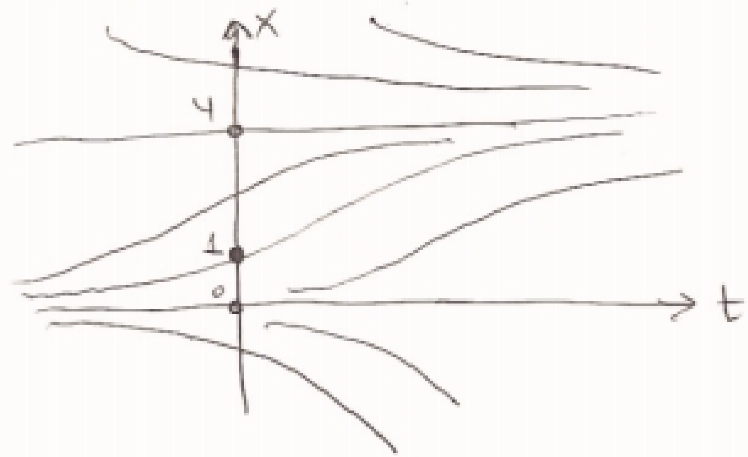
$$\frac{1}{4}\int \left(\frac{1}{x} + \frac{1}{4-x}\right) dx = \frac{1}{4}\left(\ln|x| - \ln|x-4|\right) = t + c$$

$$\left|\frac{x}{x-4}\right| = e^{4t+c} = Ae^{4t}$$

$x(0) = 1, \ 0 < 1 < x: \quad \frac{1}{3} = A$

i.e.: $\frac{x}{4-x} = \frac{1}{3}e^{4t} \Rightarrow 3x = (4-x)\cancel{\tfrac{1}{3}}e^{4t} \Rightarrow 3x + xe^{4t} = 4e^{4t}$

$$\Rightarrow x = 4e^{4t}/3 + e^{4t} = \frac{4}{3e^{-4t} + 1} \xrightarrow{t \to \infty} 4$$



3. (20pts.) Give the general solution for the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + k^2y = 0$$

and state if the system is overdamped, critically damped, or underdamped when the spring constant k has the values:

$y = e^{rt} \Rightarrow r^2 + 4r + k^2 = 0$

(a) $k = 3$.

$r^2 + 4r + 9 = 0 \Rightarrow (r+2)^2 + 5 = 0 \Rightarrow r = -2 \pm i\sqrt{5}$ underdampet

$y(t) = (A \cos\sqrt{5}t + B \sin\sqrt{5}t)e^{-2t}$

(b) $k = 2$.

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$, $r = -2$ double : critically

$y(t) = (A + Bt)e^{-2t}$

(c) $k = 1$.

$r^2 + 4r + 1 = 0 \Rightarrow (r^2 + 4r + 4) - 3 = 0$

$(r+2)^2 - (\sqrt{3})^2 = 0 \Rightarrow r = -2 \pm \sqrt{3} < 0$ overdamped

$y(t) = A e^{(2+\sqrt{3})t} + B e^{(-2-\sqrt{3})t}$

4. (15pts.) Solve the IVP

$$\phi \quad \frac{dy}{dt} + \phi \frac{t}{t^2+1} y = t \ , \ y(0) = 1 \qquad (\sqrt{t^2+1})' = \frac{1}{2} \cdot 2t \qquad \frac{t}{\sqrt{t^2+1}}$$

ϕ'

and identify the largest interval in $t_1 < t < t_2$ for which the solution is defined.

$$\phi' = \varphi \cdot \frac{t}{t^2+1} \Rightarrow (\ln \phi)' = \frac{t}{t^2+1} = \frac{1}{2} \frac{2t}{t^2+1} = \frac{1}{2} \ln(t^2+1)'$$

$$\Rightarrow \phi = \sqrt{t^2+1}$$

$$\left(\sqrt{t^2+1}\, y\right)' = t\sqrt{t^2+1} \Rightarrow (t^2+1)^{1/2} y = \frac{1}{2}\int \sqrt{t^2+1}\, d(t^2) = \frac{1}{3}(t^2+1)^{3/2} +$$

$$\frac{1}{3}(t^2+1)^{3/2}$$

$$\cancel{\frac{1}{3} \cdot \frac{3}{2} \cdot 2}\, t\, (t^2+1)^{1/2}$$

$$\boxed{y = \frac{1}{3}(t^2+1) + C(t^2+1)^{-1/2}}$$

$$y(0) = 1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$$

$$-\infty < t < \infty$$

5. (5pts.) Compute the Wronskian of the functions $y_1(x) = \sin 2x$ and $y_2(x) = \cos x \sin x$. Are these linearly independent?

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos x \sin x = \frac{1}{2}\sin 2x \\ 2\cos 2x & \cos 2x \end{vmatrix} = 0$$

dependent

6. (20.) Solve the IVP using Laplace transforms:

$$\frac{d^2y}{dt^2} - y = e^{-t}\sin t \ , \ y(0) = y'(0) = 0 \ .$$

$$(s^2-1)Y = \frac{1}{(s+1)^2+1}$$

$$Y(s) = \frac{1}{(s-1)(s+1)\left[(s+1)^2+1\right]} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+2s+2}$$

$$A = \frac{1}{(s+1)(s^2+2s+2)}\Big|_{s=1} = \frac{1}{10}$$

$$B = \frac{1}{(s-1)(s^2+2s+2)}\Big|_{s=-1} = \frac{1}{-2} = -\frac{1}{2}$$

$$\frac{1}{(s^2-1)(s^2+2s+2)} = \frac{1}{10}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{Cs+D}{s^2+2s+2}$$

$$\therefore \quad Cs+D = \frac{1}{s^2-1} - \frac{1}{10}\frac{s^2+2s+2}{s-1} + \frac{1}{2}\frac{s^2+2s+2}{s+1}$$

$s=0$: $\quad D = -1 + \frac{2}{10} + 1 = \frac{1}{5}$

$s=2$: $\quad 2C + \frac{1}{5} = \frac{1}{3} - 1 + \frac{5}{3} = 1$

$$\Rightarrow 2C = \frac{4}{5} \Rightarrow C = \frac{2}{5}$$

i.e. $\quad Y(s) = \frac{1}{10}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{5}\frac{2s+1}{(s+1)^2+1}$

$$\frac{1}{5}\frac{2(s+1)-1}{(s+1)^2+1}$$

$$\downarrow$$

$$\boxed{y(t) = \frac{1}{10}e^{t} - \frac{1}{2}e^{-t} + \frac{2}{5}e^{-t}\cos t - \frac{1}{5}e^{-t}\sin t}$$

$y(0) = \frac{1}{10} - \frac{1}{2} + \frac{2}{5} = 0$, $\quad y'(0) = \frac{1}{10} + \frac{1}{2} - \frac{2}{5} - \frac{1}{5} = 0$ ok

$1+5-4-2=0$