Solutions, 316-XIV

March 13, 2003

1 Problem 4.9.2

Find the general solution by variation of parameters for the ode

$$y'' + y = \sec t \ .$$

Solution:
The homogeneous solution can be written as

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t) := C_1 \cos t + C_2 \sin t \ .$$

Introducing the unknown functions $u_1(t)$, $u_2(t)$ as usual, we are led to the system:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= \sec t \end{aligned}$$

or, substituting $y_1(t) = \cos t$, $y_2(t) = \sin t$

$$\begin{aligned} u_1' \cos t + u_2' \sin t &= 0 \\ -u_1' \sin t + u_2' \cos t &= \sec t \end{aligned}$$

and solving, using $W(y_1, y_2) = 1$:

$$u_1' = -\sec t \sin t = -\tan t \ , \ u_2' = \sec t \cos t = 1 \ .$$

Integrating:

$$\begin{aligned} u_1(t) &= \log|\cos t| \\ u_2(t) &= t \end{aligned}$$

Finally, putting it all together:

$$y_g(t) = y_h(t) + \log|\cos t| \cos t + t \sin t \ .$$

2 Problem 4.9.3

Find the general solution by variation of parameters for the ode

$$2x'' - 2x' - 4x = 2e^{3t} \ .$$

Solution:
To get the equation into canonical form, divede by 2 to get: $x'' - x' - 2x = e^{3t}$. The characteristic equation is

$$r^2 - r - 2 = 0 \Rightarrow r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 * 1 * (-2)}}{2 * 1} = \frac{1}{2} \pm 32 = 2, -1$$

The homogeneous solution can be written as

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t) := C_1 e^{2t} + C_2 e^{-t} \ .$$

Introducing the unknown functions $u_1(t)$, $u_2(t)$ as usual, we are led to the system:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= e^{3t} \end{aligned}$$

or, substituting $y_1(t) = e^{2t}$, $y_2(t) = e^{-t}$

$$\begin{aligned} u_1' e^{2t} + u_2' e^{-t} &= 0 \\ 2u_1' e^{2t} - u_2' e^{-t} &= e^{3t} \end{aligned}$$

and solving, using $W(y_1, y_2) = -e^{2t}e^{-t} - 2e^{2t}e^{-t} = -3e^t$:

$$u_1' = -\frac{e^{-t}e^{3t}}{-3e^t} = \frac{1}{3}e^t \ , \ u_2' = \frac{e^{2t}e^{3t}}{-3e^t} = -\frac{1}{3}e^{4t} \ .$$

Integrating:

$$\begin{aligned} u_1(t) &= \frac{1}{3}e^t \\ u_2(t) &= -\frac{1}{12}e^{4t} \end{aligned}$$

Finally, putting it all together:

$$y_g(t) = y_h(t) + \frac{1}{3}e^t e^{2t} - \frac{1}{12}e^{4t}e^{-t} = y_h(t) + \frac{1}{4}e^{3t} \ .$$

3 Problem 4.9.6

Find the general solution by variation of parameters for the ode

$$y'' + 2y' + y = e^{-t} \ .$$

Solution:

The characteristic equation is

$$r^2 + 2r + 1 = 0 \Rightarrow r = \frac{-(2) \pm \sqrt{(2)^2 - 4 * 1 * (1)}}{2 * 1} = -1(\text{double root})$$

The homogeneous solution can be written as

$$y_h(t) = C_1 y_1(t) + C_2 y_2(t) := C_1 e^{-t} + C_2 t e^{-t} \ .$$

Introducing the unknown functions $u_1(t)$, $u_2(t)$ as usual, we are led to the system:

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= e^{-t} \end{aligned}$$

or, substituting $y_1(t) = e^{-t}$, $y_2(t) = te^{-t}$ and $y_1'(t) = -e^{-t}$, $y_2'(t) = (1-t)e^{-t}$

$$\begin{aligned} u_1' e^{-t} + u_2' t e^{-t} &= 0 \\ -u_1' e^{-t} + u_2'(1-t) e^{-t} &= e^{-t} \end{aligned}$$

or, simplifying:

$$\begin{aligned} u_1' + u_2' t &= 0 \\ -u_1' + u_2'(1-t) &= 1 \end{aligned}$$

and solving, $(W(y_1, y_2) = (1-t)e^{-t}e^{-t} + te^{2t}e^{-t} = e^{-2t}$ but we don't need it!):

$$u_1' = -t \ , \ u_2' = 1 \ .$$

Integrating:

$$\begin{aligned} u_1(t) &= -\frac{t^2}{2} \\ u_2(t) &= t \end{aligned}$$

Finally, putting it all together:

$$y_g(t) = y_h(t) - \frac{t^2}{2} e^{-t} + t^2 e^{-t} = y_h(t) + \frac{t^2}{2} e^{-t} \ .$$

4 Problem 4.9.26

Find the general solution by variation of parameters for the ode

$$xy'' + (1-2x)y' + (x-1)y = xe^x \ .$$

given the homogeneous solutions

$$y_1 = e^x \ , \ y_2 = e^x \log x \ .$$

Solution:
The homogeneous solution can be written as

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x) := C_1 e^x + C_2 \log x e^x \ .$$

Introducing the unknown functions $u_1(x)$, $u_2(x)$ as usual, and after dividing the ODE by x, the coefficient of the leading term, to bring to standard form:

$$y'' + \frac{1-2x}{x} y' + \frac{x-1}{x} y = e^x \ .$$

we are led to the system

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= e^x \end{aligned}$$

or, substituting $y_1() = e^x$, $y_2(t) = \log x e^x$ and $y_1'(x) = e^x$, $y_2'(x) = (\log x + 1/x)e^x$

$$\begin{aligned} u_1' e^x + u_2' \log x e^x &= 0 \\ -u_1' e^x + u_2' \left(\log x + \frac{1}{x}\right) e^x &= e^x \end{aligned}$$

or, simplifying:

$$\begin{aligned} u_1' + u_2' \log x &= 0 \\ u_1' + u_2' \left(\log x + \frac{1}{x}\right) &= 1 \end{aligned}$$

and solving,

$$u_1' = -x \log x \ , \ u_2' = x \ .$$

Integrating:

$$\begin{aligned} u_1(t) &= -\frac{x^2}{2}\left(\log x - \frac{1}{2}\right) \\ u_2(t) &= \frac{x^2}{2} \end{aligned}$$

Finally, putting it all together:

$$y_g(t) = y_h(t) - \frac{x^2}{2}\left(\log x - \frac{1}{2}\right)e^x + \frac{x^2}{2}\log x e^x = y_h(t) + \frac{x^2}{2}e^x \ .$$