Solutions, 316-XVII

March 27, 2003

7.7¡5*,6,7,9,10*,23*,26,27¿

1 Problem 7.7.5

Determine the inverse Laplace transform using convolutions:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$$

Solution:

$$\begin{aligned}\frac{1}{s(s^2+1)} &= \frac{1}{s}\frac{1}{s^2+1} \Rightarrow \\ \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= 1 \star \sin t = \int_0^t \sin\tau d\tau \\ &= -\cos\tau|_0^t = 1-\cos t\end{aligned}$$

(*Of course this was obvious from basics:* $(1/s)F(s)$ *is the transform of the integral* $\int_0^\infty f(\tau)d\tau$).

2 Problem 7.5.6

Determine the inverse Laplace transform using convolutions:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$

Solution:

$$\begin{aligned}
\frac{1}{(s+1)(s+2)} &= \frac{1}{s+1}\frac{1}{s+2} \Rightarrow \\
\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} &= e^{-t}\star e^{-2t} = \int_0^t e^{-(t-\tau)}e^{-2\tau}d\tau \\
&= e^{-t}\int_0^t e^{\tau-2\tau}d\tau \\
&= e^{-t}\int_0^t e^{-\tau}d\tau \\
&= -e^{-t}\,e^{-\tau}\Big|_0^t = -e^{-2t}+e^{-t}
\end{aligned}$$

3 Problem 7.7.10

Determine the inverse Laplace transform using convolutions:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$$

Solution:

$$\begin{aligned}
\frac{1}{s^3(s^2+1)} &= \frac{1}{s^3}\frac{1}{s^2+1} \Rightarrow \\
\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \frac{1}{2}t^2 \star \sin t = \int_0^t \frac{1}{2}(t-\tau)^2 \sin\tau d\tau \\
&= -\frac{1}{2}\int_0^t (t-\tau)^2 d\cos\tau \\
&= -\frac{1}{2}\,(t-\tau)^2\cos\tau\Big|_0^t + \frac{1}{2}\int_0^t \cos\tau d(t-\tau)^2 \\
&= \frac{1}{2}t^2 - \int_0^t (t-\tau)d\sin\tau \\
&= \frac{1}{2}t^2 - \,(t-\tau)\sin\tau|_0^t - \int_0^t \sin\tau d\tau \\
&= \frac{1}{2}t^2 + \,\cos\tau|_0^t \\
&= \frac{1}{2}t^2 + \cos t - 1
\end{aligned}$$

4 Problem 7.7.23

Find the transfer function $H(s)$, the impulse response function, $h(t)$ and give a formula for the solution of the IVP:

$$y'' + 9y = g(t) \ ; \ y(0) = 2 \ , \ y'(0) = -3 \ .$$

Solution:

$$\begin{aligned} (s^2+9)Y(s) &= y(0)s + y'(0) + G(s) \\ Y(s) &= y(0)sH(s) + y'(0)H(s) + G(s)H(s) \\ &= 2sH(s) - 3H(s) + G(s)H(s) \end{aligned}$$

where

$$\begin{aligned} H(s) &= \frac{1}{s^2+9} \\ h(t) &= \cos 3t \end{aligned}$$

Since

$$\mathcal{L}^{-1}\{sH(s)\} = h'(t)$$

we have

$$y(t) = 2h'(t) - 9h(t) + h(t) \star g(t) = -6\sin 3t - 9\cos 3t + \int_0^t h(t-\tau)g(\tau)d\tau$$

or

$$y(t) = -6\sin 3t - 9\cos 3t + \int_0^t \cos 3(t-\tau)g(\tau)d\tau \ .$$

5 Problem 7.7.26

$$y'' + 2y' - 15y = g(t) \ ; \ y(0) = 0 \ , \ y'(0) = 8 \ .$$

Solution:

$$\begin{aligned} (s^2+2s-15)Y(s) &= y(0)(s+2) + y'(0) + G(s) \\ Y(s) &= y(0)(s+2)H(s) + y'(0)H(s) + G(s)H(s) \\ &= 8H(s) + G(s)H(s) \end{aligned}$$

where

$$\begin{aligned} H(s) &= \frac{1}{s^2+2s-15} = \frac{1}{(s+1)^2-4^2} = \frac{1}{(s-3)(s+5)} \\ &= \frac{A}{s-3} + \frac{B}{s+5} \end{aligned}$$

with

$$\begin{aligned} A &= \lim_{s\to 3}(s-3)H(s) \\ &= \left.\frac{1}{s+5}\right|_{s=3} \\ &= \frac{1}{8} \end{aligned}$$

and

$$\begin{aligned} B &= \lim_{s\to -5}(s+5)H(s) \\ &= \left.\frac{1}{s-3}\right|_{s=-5} \\ &= \frac{-1}{8} \end{aligned}$$

so that

$$h(t) = \frac{1}{8}\left(e^{3t} - e^{-5t}\right) \ .$$

Then, a formula for $y(t)$ can be found using convolutions:

$$y(t) = 8h(t) + h(t) \star g(t) = 8h(t) + \int_0^t h(t-\tau)g(\tau)d\tau$$

or

$$y(t) = e^{3t} - e^{-5t} + \frac{e^{3t}}{8}\int_0^t e^{-3\tau}g(\tau)d\tau - \frac{e^{-5t}}{8}\int_0^t e^{5\tau}g(\tau)d\tau \ .$$

	t-domain ($f(t)$)	s-domain ($F(s)$)
1	$f(t)$	$F(s)$
2	$C_1 f_1(t) + C_2 f_2(t)$	$C_1 F_1(s) + C_2 F_2(s)$
3	1	$\frac{1}{s}$
4	t	$\frac{1}{s^2}$
5	t^n	$\frac{n!}{s^{n+1}}$
6	e^{at}	$\frac{1}{s-a}$
7	$e^{at} f(t)$	$F(s-a)$
8	$\cos bt$	$\frac{s}{s^2+b^2}$
9	$\sin bt$	$\frac{b}{s^2+b^2}$
10	$f'(t)$	$sF(s) - f(0)$
11	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
12	$tf(t)$	$-F'(s)$
13	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
14	$\int_0^t g(\tau) h(t-\tau) d\tau$	$G(s)H(s)$
15	$\int_0^t g(\tau) d\tau$	$\frac{1}{s} G(s)$

Table 1: *Useful Laplace transforms*