Solutions, 316-XXV

April 28, 2003

25(4/22) Phase plane and stability
12.2¡1*,2,3*,4,5,7*,8,9,13*¿
CAUTION: there may be errors!!!

1 Problem 12.2.1

Classify the critical point at the origin for the system

$$\begin{aligned}\frac{dx}{dt} &= 5x+6y \\ \frac{dy}{dt} &= -5x-8y\end{aligned}$$

Solution:

$$\begin{aligned}\det\begin{pmatrix} 5-r & 6 \\ -5 & -8-r\end{pmatrix} = 0 &\Rightarrow (r+8)(r-5)+30=0 \Rightarrow \\ r^2+3r-40=0 &\Rightarrow (r+5)(r-2)=0\end{aligned}$$

so that the eigenvalues are

$$r_1 = 2\ ,\ r_2 = -5$$

and the origin is an unstable saddle point.

2 Problem 12.2.3

Classify the critical point at the origin for the system

$$\begin{aligned} \frac{dx}{dt} &= 3x + 5y \\ \frac{dy}{dt} &= -5x - 5y \end{aligned}$$

Solution:

$$\det \begin{pmatrix} 3-r & 5 \\ -5 & -5-r \end{pmatrix} = 0 \;\Rightarrow\; (r+5)(r-3) + 25 = 0 \Rightarrow$$
$$r^2 + 2r + 10 = 0 \;\Rightarrow\; (r+1)^2 + 9 = 0$$

so that the eigenvalues are

$$r_1 = -1 + 3i \;,\; r_2 = -1 - 3i$$

and the origin is a stable spiral: all solutions tend to zero as $t \to \infty$.

3 Problem 12.2.5

Classify the critical point at the origin for the system

$$\begin{aligned} \frac{dx}{dt} &= -6x + 3y \\ \frac{dy}{dt} &= x - 4y \end{aligned}$$

Solution:

$$\det \begin{pmatrix} -6-r & 3 \\ 1 & -4-r \end{pmatrix} = 0 \;\Rightarrow\; (r+6)(r+4) - 3 = 0 \Rightarrow$$
$$r^2 + 10r + 21 = 0 \;\Rightarrow\; (r+7)(r+3) = 0$$

so that the eigenvalues are

$$r_1 = -3 \ , \ r_2 = -7$$

and the origin is a stable improper node: all solutions tend to zero as $t \to \infty$.

4 Problem 12.2.7

Find and classify the critical point of the linear system

$$\begin{aligned}\frac{dx}{dt} &= -4x + 2y + 8 \\ \frac{dy}{dt} &= x - 2y + 1\end{aligned}$$

Solution:
First we set the rhs equal to zero to determine the critical point:

$$\left.\begin{aligned}-4x + 2y &= -8 \\ x - 2y &= -1\end{aligned}\right\} \Rightarrow \quad \left.\begin{aligned}-3x = -9 &\Rightarrow x = 1 \\ 2y = x + 1 &\Rightarrow y = 1\end{aligned}\right\}$$

So, the critical point is $(x, y) = (1, 1)$. We introduce variables

$$u = x - 1 \ , \ v = y - 1 \ .$$

In terms of (u, v) the system becomes

$$\begin{aligned}\frac{du}{dt} &= -4(u+1) + 2(v+1) + 8 = -4u + 2v \\ \frac{dv}{dt} &= (u+1) - 2(v+1) + 1 = u - 2v\end{aligned}$$

We find now the eigenvalues:

$$\det\begin{pmatrix} -4-r & 2 \\ 1 & -2-r \end{pmatrix} = 0 \Rightarrow (r+2)(r+4) - 2 = 0 \Rightarrow$$
$$r^2 + 6r + 6 = 0 \Rightarrow (r+3)^2 - 3 = 0$$

so that the eigenvalues are

$$r_1 = -3 + \sqrt{3} \ , \ r_2 = -3 - \sqrt{3}$$

and the point (1,1) is a stable improper node: all solutions tend to it as $t \to \infty$.

5 Problem 12.2.8

Find and classify the critical point of the linear system

$$\begin{aligned}\frac{dx}{dt} &= -3x - y - 4 \\ \frac{dy}{dt} &= 29x + y + 30\end{aligned}$$

Solution:
First we set the rhs equal to zero to determine the critical point:

$$\left.\begin{aligned}-3x - y &= 4 \\ 29x + y &= -30\end{aligned}\right\} \Rightarrow \left.\begin{aligned}26x = -26 \quad &\Rightarrow \; x = -1 \\ y = -3x - 4 = -1 \quad &\end{aligned}\right\}$$

So, the critical point is $(x, y) = (-1, -1)$. We introduce variables

$$u = x + 1 \ , \ v = y + 1 \ .$$

In terms of (u, v) the system becomes

$$\begin{aligned}\frac{du}{dt} &= -3(u-1) - (v-1) - 4 = -3u - v \\ \frac{dv}{dt} &= 29(u-1) + (v-1) + 30 = 29u + v\end{aligned}$$

We find now the eigenvalues:

$$\det\begin{pmatrix} -3-r & -1 \\ 29 & 1-r \end{pmatrix} = 0 \ \Rightarrow \ (r-1)(r+3) + 29 = 0 \Rightarrow$$
$$r^2 + 2r + 26 = 0 \ \Rightarrow \ (r+1)^2 + 25 = 0$$

so that the eigenvalues are

$$r_1 = -1 + 5i \ , \ r_2 = -1 - 5i$$

and the point (-1,-1) is a stable spiral: all solutions tend to it as $t \to \infty$.

6 Problem 12.2.13

Classify the critical point at the origin for the system and sketch the phase plane diagram

$$\begin{aligned} \frac{dx}{dt} &= x - 6y \\ \frac{dy}{dt} &= 2x - 7y \end{aligned}$$

Solution:
We find the eigenvalues:

$$\det \begin{pmatrix} 1-r & -6 \\ 2 & -7-r \end{pmatrix} = 0 \;\Rightarrow\; (r+7)(r-1)+12 = 0 \Rightarrow$$
$$r^2 + 6r + 5 = 0 \;\Rightarrow\; (r+5)(r+1) = 0$$

so that the eigenvalues are

$$r_1 = -1 \;,\; r_2 = -5$$

and the origin is an unstable saddle point. Now the eigenvectors:

$$\mathbf{A}\mathbf{u}_i = r_i \mathbf{u}_i \;:$$

or

$$\begin{pmatrix} 1-r & -6 \\ 2 & -7-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

1. Eigenvalue $r_1 = -1$:

   $$\begin{aligned} 2x &- 6y = 0 \,, \\ 2x &- 6y = 0 \,. \end{aligned}$$

   The solution is

   $$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} .$$

2. Eigenvalue $r_2 = -5$:

   $$\begin{aligned} 6x &- 6y = 0 \,, \\ 2x &- 2y = 0 \,. \end{aligned}$$

   The solution is

   $$\mathbf{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

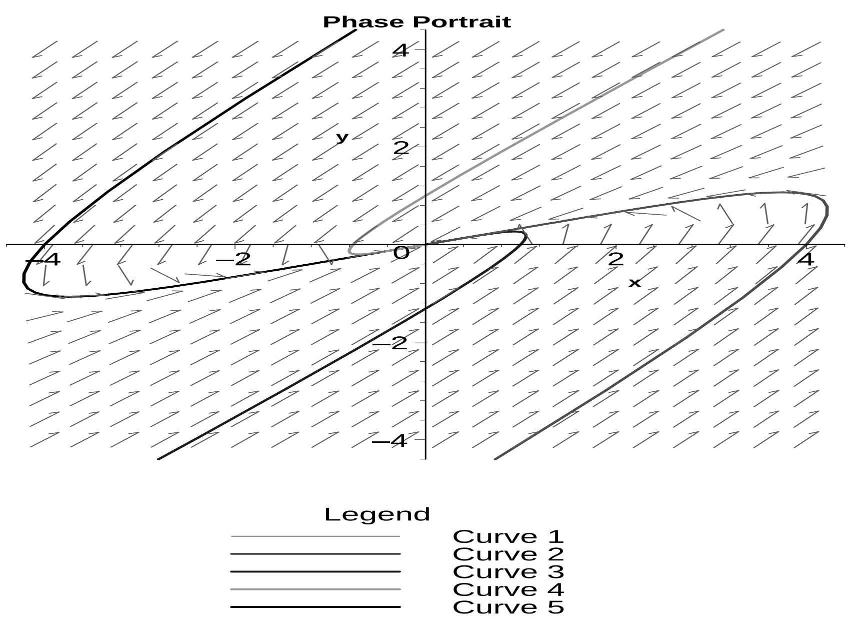
```
> restart:
> with(DEtools): with(plots): with(plottools):

Warning, the name changecoords has been redefined

Warning, the names arrow and translate have been redefined
```

In this example, we are looking at x'=y,y'=-x(1+y), with two sets of initial conditions: (x(0),y(0))=(1,0), (x(0),y(0))=(2,0).

```
> Digits := 10:
> plot1 :=
DEplot([diff(x(t),t)=x(t)-6*y(t),diff(y(t),t)=2*x(t)-7*y(t)],
[x(t),y(t)], t=-5..5, scene=[x(t),y(t)],
[[x(0)=1,y(0)=0],[x(0)=-4,y(0)=0],[x(0)=0,y(0)=1],[x(0)=4,y(0)=0]],
x=-4..4, y=-4..4, linecolor=[red,blue,green,black], arrows=SMALL,
method=rkf45, stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD, 10]);
```



For a plot of x(t) only....

7 Problem 12.2.15

Classify the critical point at the origin for the system and sketch the phase plane diagram

$$\begin{aligned} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 5x - 2y \end{aligned}$$

Solution:
We find the eigenvalues:

$$\det \begin{pmatrix} 1-r & 2 \\ 5 & -2-r \end{pmatrix} = 0 \;\Rightarrow\; (r+2)(r-1) - 10 = 0 \Rightarrow$$
$$r^2 + r - 12 = 0 \;\Rightarrow\; (r+4)(r-3) = 0$$

so that the eigenvalues are

$$r_1 = 3\ ,\ r_2 = -4$$

and the origin is an unstable saddle point. Now the eigenvectors:

$$\mathbf{A}\mathbf{u}_i = r_i \mathbf{u}_i \;\; :$$

or

$$\begin{pmatrix} 1-r & 2 \\ 5 & -2-r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

1. Eigenvalue $r_1 = 3$:

$$\begin{aligned} -2x + 2y &= 0\ , \\ 5x - 5y &= 0\ . \end{aligned}$$

   The solution is

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

2. Eigenvalue $r_2 = -4$:

$$\begin{aligned} 5x + 2y &= 0\ , \\ 5x + 2y &= 0\ . \end{aligned}$$

   The solution is

$$\mathbf{u}_2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix} .$$

```
> restart:
> with(DEtools): with(plots): with(plottools):
```

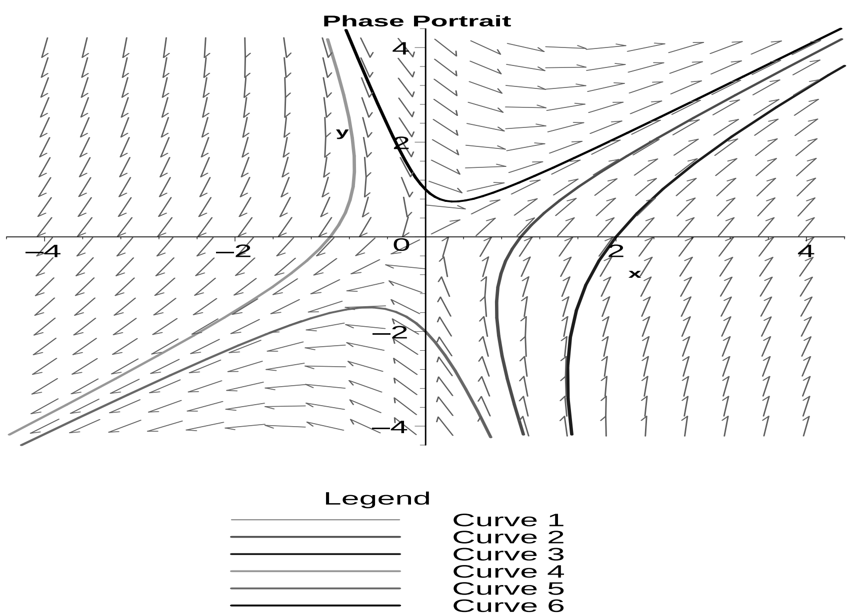
```
Warning, the name changecoords has been redefined

Warning, the names arrow and translate have been redefined
```

In this example, we are looking at x'=x+2y,y'=5x-2y, with five sets of initial conditions:

(x(0),y(0))=(1,0),
(x(0),y(0))=(2,0),(x(0),y(0))=(-1,0),(x(0),y(0))=(0,1),(x(0),y(0))=(0, -2).

```
> Digits := 10:
> plot1 :=
DEplot([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=5*x(t)-2*y(t)],
[x(t),y(t)], t=-5..5, scene=[x(t),y(t)],
[[x(0)=1,y(0)=0],[x(0)=2,y(0)=0],[x(0) = -1,y(0)=0],[x(0) =
0,y(0)=1],[x(0)=0,y(0)=-2]], x=-4..4, y=-4..4,
linecolor=[red,blue,green,magenta,black], arrows=SMALL, method=rkf45,
stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD, 10]);
```



For a plot of x(t) only....

```
> plot2 :=
DEplot([diff(x(t),t)=2*x(t)+y(t),diff(y(t),t)=5*x(t)-2*y(t)],
[x(t),y(t)], t=-5..5, scene=[t,x(t)],
[[x(0)=1,y(0)=0],[x(0)=2,y(0)=0],[x(0) = -1,y(0)=0],[x(0) =
0,y(0)=1],[x(0)=0,y(0)=-2]], x=-4..4, y=-4..4,
linecolor=[red,blue,green,magenta,black], arrows=SMALL, method=rkf45,
stepsize=0.05):
> display({plot2}, title="Solution Curve", titlefont = [HELVETICA,
BOLD, 12], labels = ["t","x"], labelfont = [HELVETICA, BOLD, 10]);
```