Solutions, 316-XXVII

May 6, 2003

27(4/29) More Phase Plane; energy 12.4¡1,2*,7,8,10,13*,14,17*)
CAUTION: there may be errors!!!

1 Problem 12.4.2

Find the potential energy function $G(x)$ and the energy function $E(x,v)$ for the given equations. Select E so that $E(0,0) = 0$.

$$\frac{d^2x}{dt^2} + \cos x = 0 \ . \tag{1}$$

Solution:
As usual, multiplying through by dx/dt and integrating:

$$\begin{aligned} \frac{dx}{dt}\left\{\frac{d^2x}{dt^2} + \cos x\right\} &= 0 \\ \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \sin x\right\} &= 0 \\ \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \sin x &= E \ , \text{ a constant} \end{aligned}$$

or, introducing the potential energy

$$G(x) := \sin x \ ,$$

we have

$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + G(x) = E(x, \frac{dx}{dt}) \ .$$

2 Problem 12.4.13

Use the energy function to assist in drawing the phase plane diagram for the given non-conservative system

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x - x^3 = 0 \ . \tag{2}$$

Solution:
Multiplying through by dx/dt and integrating:

$$\begin{aligned} \frac{dx}{dt}\left\{\frac{d^2x}{dt^2} + x - x^3\right\} &= -\left(\frac{dx}{dt}\right)^2 \\ \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4\right\} &= -\left(\frac{dx}{dt}\right)^2 \end{aligned}$$

so that if we define the energy as

$$E := \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$$

then the DE can be rewritten to exhibit the energy and the dissipation term:

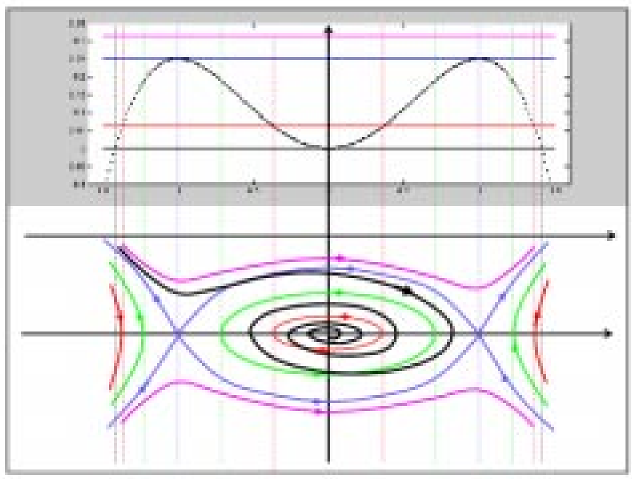
$$\frac{dE(x, \frac{dx}{dt})}{dt} = -\left(\frac{dx}{dt}\right)^2 \ .$$

Clearly, the term on the right is never positive, so the solution will always head towards the direction of decreasing energy.

We first ignore the dissipation term and draw the phase portrait of the corresponding conservative system. Then the actual orbits will follow paths in the direction of decreasing energy. Plot $G(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$ together with several energy "levels". Given a value of E, a phase orbit with that energy can only exist for positions where $G(x) \leq E$. Since the difference

$$E - G(x) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2$$

equals the kinetic energy, it is never negative (or the velocity would become imaginary!).



Only the range of x-values for which the potential energy is less than the total energy are accessible. So, assume

$$E(x_0, x_0') = E_0 \ .$$

Then, since

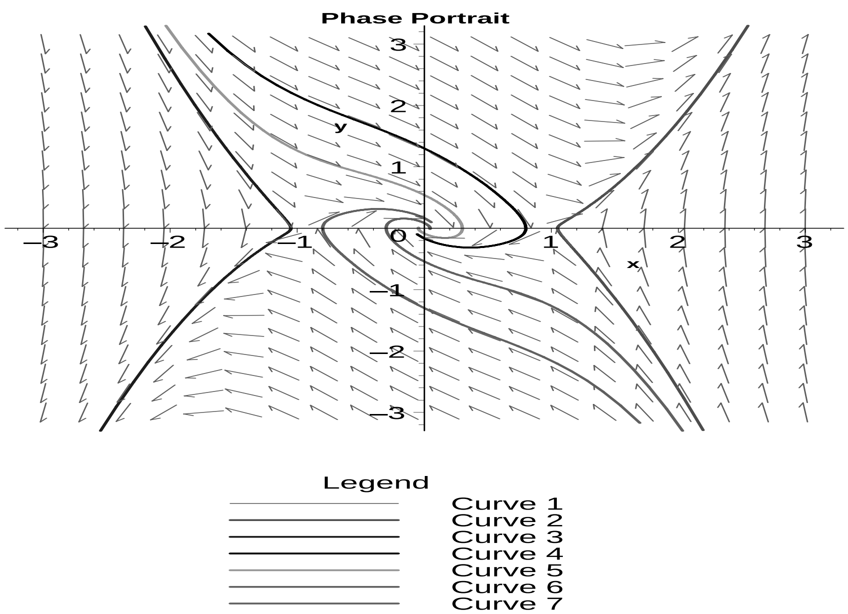
$$\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + G(x) = E_0$$

we have that

$$G(x) = E_0 - \frac{1}{2}\left(\frac{dx}{dt}\right)^2 \leq E_0 \ .$$

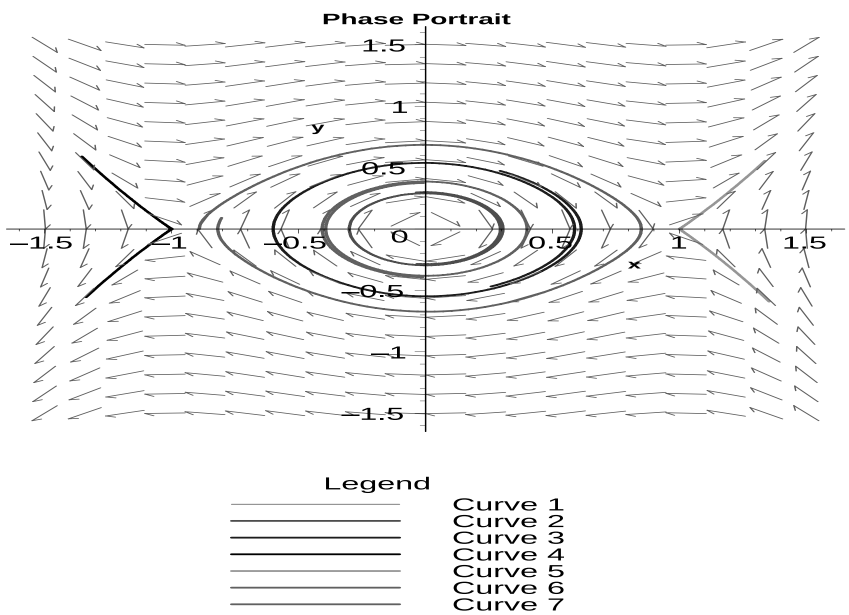
```
> restart:
> with(DEtools): with(plots): with(plottools):

> Digits := 10:
> plot1 :=
> DEplot([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)*(1-x(t)^2)-y(t)],
> [x(t),y(t)], t=-4..4, scene=[x(t),y(t)],
> [[x(0)=-.8,y(0)=0],[x(0)=-.3,y(0)=0],[x(0)=.3,y(0)=0],[x(0)=.8,y(0)=0]
> ,[x(0)=1.05,y(0)=0],[x(0)=-1.05,y(0)=0]], x=-3..3, y=-3..3,
> linecolor=[red,blue,black,green,orange,orange], arrows=SMALL,
> method=rkf45, stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



What the phase plot really looks like

```
> plot1 :=
> DEplot([diff(x(t),t)=y(t),diff(y(t),t)=-x(t)*(1-x(t)^2)-.01*y(t)],
> [x(t),y(t)], t=-5..5, scene=[x(t),y(t)],
> [[x(0)=-.6,y(0)=0],[x(0)=-.3,y(0)=0],[x(0)=.4,y(0)=0],[x(0)=.85,y(0)=0
> ],[x(0)=1.0005,y(0)=0],[x(0)=-1.0005,y(0)=0]], x=-1.5..1.5,
> y=-1.5..1.5, linecolor=[red,blue,black,green,orange,orange],
> arrows=SMALL, method=rkf45, stepsize=0.05):
> display({plot1}, title="Phase Portrait", titlefont = [HELVETICA,
> BOLD, 12], labels = ["x","y"], labelfont = [HELVETICA, BOLD,
10]);
```



What the phase plot would look like if the dissipation term had a coefficient equal to .01 (instead of 1).

3 Problem 12.4.17

Nonlinear Spring. The general nonlinear spring equation

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = 0 \ , \tag{3}$$

where $\alpha > 0$ and β are parameters, is used to model a variety of physical phenomena. Holding $\alpha > 0$ fixed, sketch the potential function and the phase plane diagram for $\beta > 0$ and also for $\beta < o$. Describe how the behavior of solutions to the equation differs in these two cases.

Solution:

Multiplying through by dx/dt and integrating:

$$\begin{aligned} \frac{dx}{dt}\left\{\frac{d^2x}{dt^2} + \alpha x + \beta x^3\right\} &= 0 \\ \frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4\right\} &= 0 \end{aligned}$$

so that if we define the energy as

$$E := \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4$$

then the DE becomes

$$\frac{dE(x, \frac{dx}{dt})}{dt} = 0 \ .$$

where

$$E(x, \frac{dx}{dt}) = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + G(x)$$

with the potential energy $G(x)$ defined as:

$$G(x) = \frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 \ .$$

Refer to previous problem for plot with $\beta < 0$. For $\beta > 0$ the origin is a center and all orbits are closed curves about the origin.