Solutions, 316-IV

February 4, 2003

1 Problem 2.3.18

Solve the IVP:

$$\frac{dy}{dx} + 4y - e^{-x} = 0 \ , \ y(0) = \frac{4}{3}$$

Solution:
The integrating factor, $\mu(x)$, is

$$\mu(x) = e^{\int 4dx} = e^{4x}$$

so that

$$\begin{aligned}
e^{4x}\left(\frac{dy}{dx} + 4y\right) &= e^{-x} * e^{4x} = e^{3x} \\
\frac{d}{dx}\left(ye^{4x}\right) &= e^{3x} \\
ye^{4x} &= \int e^{3x}dx + C \\
ye^{4x} &= \frac{1}{3}e^{3x} + C \\
y(x) &= \frac{1}{3}e^{-x} + Ce^{-4x} \\
y(0) &= \frac{1}{3} + C = \frac{4}{3} \to C = 1
\end{aligned}$$

Then the solution to the IVP is:

$$y(x) = \frac{1}{3}e^{-x} + e^{-4x} \ .$$

2 Problem 2.3.23

In Ex. 2 we have that Ra^1 decays into Ra^2 at the rate of $40e^{-20t}$kg/sec and the decay constant for Ra^2 is $k = 5$/sec. I.e. the equation describing the total amount of Ra^2 present, $y(t)$ is:

$$\frac{dy}{dt} + 5y = 40e^{-20t} \ .$$

Find $y(t)$ for $t \geq 0$ if $y(0) = 10$kg.
Solution: The integrating factor, $\mu(t)$, is

$$\mu(t) = e^{\int 5dt} = e^{5t}$$

so that

$$\begin{aligned}
e^{5t}\left(\frac{dy}{dt} + 5y\right) &= 40e^{-20t} * e^{5t} = 40e^{-15x} \\
\frac{d}{dt}\left(ye^{5t}\right) &= 40e^{-15t} \\
ye^{5t} &= 40\int e^{-15t}dt + C \\
ye^{5t} &= -\frac{40}{15}e^{-15t} + C \\
y(t) &= -\frac{40}{15}e^{-20t} + Ce^{-5t} \\
y(0) &= -\frac{8}{3} + C = 10 \rightarrow C = \frac{38}{3} = 12.67
\end{aligned}$$

Then the solution to the IVP is:

$$y(t) = -\frac{8}{3}e^{-20t} + \frac{38}{3}e^{-5t} \ .$$

3 Problem 2.6.30

Solve the differential equation:

$$(x + y - 1)dx + (y - x - 5)dy = 0 \ .$$

Solution: We solve this problem in 3 steps:

1. Change to (u, v) variables such that the constants are removed: $u = x - x_0$, $v = y - y_0$ so that

$$x + y - 1 = u + v = (x - x_0) + (y - y_0) = x + y - (x_0 + y_0)$$

and

$$y - x - 5 = -u + v = -(x - x_0) + (y - y_0) = -x + y + (x_0 - y_0)$$

so that we get the system

$$\begin{aligned} x_0 + y_0 &= 1 \\ x_0 - y_0 &= -5 \end{aligned}$$

with solution: $x_0 = -2$, $y_0 = 3$, i.e. $u = x + 2$, $v = y - 3$. In terms of u, v the equation becomes:

$$(u + v)du + (v - u)dv = 0 \ .$$

2. We now write with u and v as independent and dependent variables, respectively. We have:

$$\frac{dv}{du} = \frac{u + v}{u - v} \ .$$

This equation can be solved by the substitution

$$v = uz \ \Rightarrow \frac{dv}{du} = z + u\frac{dz}{du} \ .$$

Substituting:

$$\frac{dv}{du} = z + u\frac{dz}{du} = \frac{u + v}{u - v} = \frac{1 + v/u}{1 - v/u} = \frac{1 + z}{1 - z} \ ,$$

or

$$\begin{aligned}
z + u\frac{dz}{du} &= \frac{1+z}{1-z} \\
\frac{dz}{du} &= \frac{1}{u}\left(\frac{1+z}{1-z} - z\right) \\
\frac{dz}{du} &= \frac{1}{u}\frac{1+z-z+z^2}{1-z} \\
\int \frac{1-z}{1+z^2}dz &= \int \frac{1}{u}du + C \\
\tan^{-1} z - \frac{1}{2}\log(1+z^2) &= \log|u| + C
\end{aligned}$$

3. Recal now that $z = v/u$ so that

$$\begin{aligned}
\tan^{-1}(v/u) - \frac{1}{2}\log(1+(v/u)^2) &= \log|u| + C \\
\tan^{-1}(v/u) - \frac{1}{2}\log\frac{u^2+v^2}{u^2} &= \log|u| + C \\
\tan^{-1}(v/u) - \frac{1}{2}\log(u^2+v^2) + \frac{1}{2}\log(u^2) &= \log|u| + C \\
\tan^{-1}(v/u) - \frac{1}{2}\log(u^2+v^2) + \log|u| &= \log|u| + C \\
\tan^{-1}(v/u) &= \frac{1}{2}\log(u^2+v^2) + C\ .
\end{aligned}$$

We finally substitute back x, y:

$$\tan^{-1}\left(\frac{y-3}{x+2}\right) = \frac{1}{2}\log((x+2)^2 + (y-3)^2) + C\ .$$