Solutions, 316-VI

February 5, 2003

1 Problem 4.6.7

Find the general solution of

$$4y'' - 4y' + 26y = 0 \ .$$

Solution:

$$\begin{array}{rcl} 26y & = & 26e^{rt} \\ + & & + \\ -4y' & = & -4re^{rt} \\ + & & + \\ 4y'' & = & 4r^2e^{rt} \\ = & & = \\ 0 & = & (4r^2 - 4r + 26)e^{rt} \end{array}$$

Then, we factor

$$4r^2 - 4r + 26 \Rightarrow r_{\pm} = \mu \pm \nu i = 2 \pm 10i$$

so that the general solution is

$$\begin{aligned} y(t) &= Ae^{\mu t}\cos(\nu t) + Be^{\mu t}\sin(\nu t) \\ &= Ae^{2t}\cos(10t) + Be^{2t}\sin(10t) \end{aligned}$$

2 Problem 4.6.21

Solve the IVP

$$y'' - 2y' - 2y = 0 \ , \ y(0) = 2 \ , \ y'(0) = 1 \ .$$

Solution:

$$\begin{array}{rcl} 2y & = & 2e^{rt} \\ + & & + \\ 2y' & = & 2re^{rt} \\ + & & + \\ y'' & = & r^2 e^{rt} \\ = & & = \\ 0 & = & (r^2 + 2r + 2)e^{rt} \end{array}$$

Then, we factor

$$r^2 + 2r + 2 \Rightarrow r_\pm = -1 \pm i$$

so that the general solution is

$$\begin{aligned} y(t) &= Ae^{-t}\cos t + Be^{-t}\sin t \\ y'(t) &= Ae^{-t}(-\cos t - \sin t) + Be^{-t}(-\sin t + \cos t) \end{aligned}$$

Then

$$y(0) = 2 = A \ , \ y'(0) = 1 = -A + B \ \Rightarrow B = 3$$

giving for the solution:

$$y(t) = 2e^{-t}\cos t + 3e^{-t}\sin t$$

3 Problem 4.6.28

To see the effect of changing the parameter b in the problem

$$y'' + by' + 4y = 0 \ ; \ y(0) = 1 \ , \ y'(0) = 0$$

solve the problem for $b = 5, 4, 2$ and sketch the solutions. **Solution:**

$$\begin{array}{rcl} 4y & = & 4e^{rt} \\ + & & + \\ by' & = & bre^{rt} \\ + & & + \\ y'' & = & r^2e^{rt} \\ = & & = \\ 0 & = & (r^2+br+4)e^{rt} \end{array}$$

and factoring:

$$r_{\pm} = \frac{-b \pm \sqrt{b^2-16}}{2} = \frac{-b \pm \sqrt{(b-4)(b+4)}}{2}$$

so that the equation has two real, distinct roots for $b < -4$ or $b > 4$, two real equal roots (double root) for $b = \pm 4$ and two complex conjugate roots for $-4 < b < 4$. Therefore the solutions for the 3 cases given $(b = 5, 4, 2)$ are:

$$\begin{array}{c} b = 5 \Rightarrow r_+ = -1 \ , \ r_- = -4 \\ y(t) = Ae^{-t} + Be^{-4t} \quad ; \quad y(0) = 1 \\ y'(t) = -Ae^{-t} - 4Be^{-4t} \quad ; \quad y'(0) = 0 \\ A + B = 1 \ , \ -A - 4B = 0 \ \Rightarrow \ A = \frac{4}{3} \ , \ B = \frac{-1}{3} \\ y(t) = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t} \quad . \end{array}$$

$$\begin{array}{c} b = 4 \Rightarrow r_{\pm} = -2(\text{double root}) \\ y(t) = Ae^{-2t} + Bte^{-2t} \quad ; \quad y(0) = 1 \\ y'(t) = -2Ae^{-2t} + Be^{-2t}(1-2t) \quad ; \quad y'(0) = 0 \\ A = 1 \ , \ -2A + B = 0 \ \Rightarrow \ A = 1 \ , \ B = 2 \\ y(t) = e^{-2t} + 2te^{-2t} \quad . \end{array}$$

$$\begin{array}{c} b = 2 \Rightarrow r_{\pm} = -1 \pm i\sqrt{3} \\ y(t) = Ae^{-t}\cos\left(\sqrt{3}t\right) \quad + \quad Be^{-t}\sin\left(\sqrt{3}t\right) \ ; \ y(0) = 1 \ , \ y'(0) = 0 \end{array}$$

$$y'(t) = Ae^{-t}\left(-\cos\left(\sqrt{3}t\right) - \sqrt{3}\sin\left(\sqrt{3}t\right)\right) \quad + \quad Be^{-t}\left(-\sin\left(\sqrt{3}t\right) + \sqrt{3}\cos\left(\sqrt{3}t\right)\right)$$

$$A = 1\ ,\ -A + \sqrt{3}B = 0 \quad \Rightarrow \quad A = 1\ ,\ B = \frac{\sqrt{3}}{3}$$

$$y(t) = e^{-t}\cos\left(\sqrt{3}t\right) \quad + \quad \frac{\sqrt{3}}{3}e^{-t}\sin\left(\sqrt{3}t\right)$$

We now use Maple to plot all three functions for $0 \leq t \leq 2\pi$.

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> restart:
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$$m := 1$$
$$Om0 := 5.$$

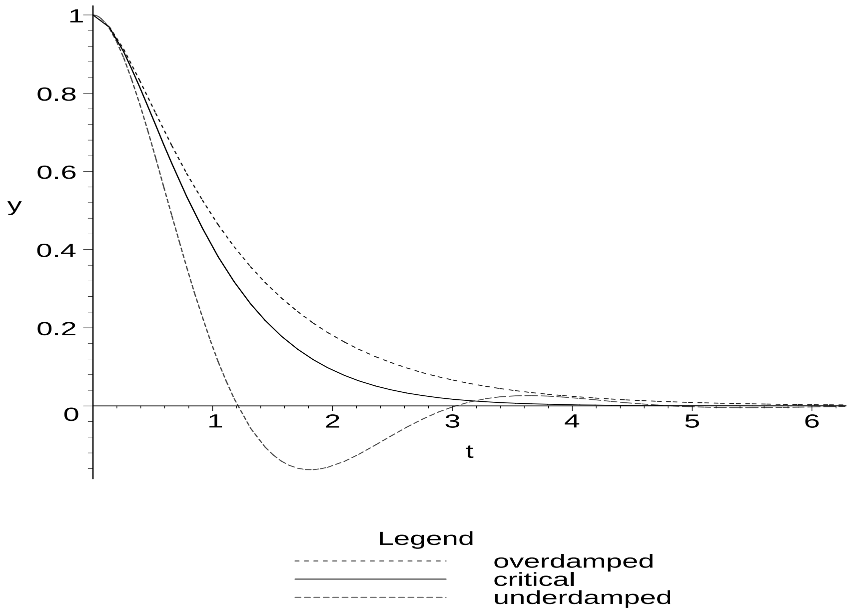
```
> Y1 := t -> (4/3)*exp(-t) - (1/3)*exp(-4*t);
```

$$Y1 := t \to \frac{4}{3}\,e^{(-t)} - \frac{1}{3}\,e^{(-4\,t)}$$

```
> Y2 := t -> exp(-2*t)*(1 + 2*t);  Y3 := t -> exp(-t)*(cos(sqrt(3)*t)
+
> (1/sqrt(3))*sin(sqrt(3)*t));
```

$$Y2 := t \to e^{(-2\,t)}\,(1 + 2\,t)$$
$$Y3 := t \to e^{(-t)}\,(\cos(\sqrt{3}\,t) + \frac{\sin(\sqrt{3}\,t)}{\sqrt{3}})$$

```
> plot([Y1(t),Y2(t),Y3(t)], t =
> 0..2*Pi,color=[blue,black,red],labels=[t,y],legend=[overdamped,critica
> l,underdamped],linestyle=[DASHDOT,SOLID,DASH]);
```



4 Problem 4.6.39

Swinging Door: The motion of a swinging door with an adjustment screw that controls the amount of friction on the hinges is governed by the initial value problem

$$I\theta'' + b\theta' + k\theta = 0 \ ; \ \theta(0) = \theta_0 \ , \ \theta'(0) = u_0 \ ,$$

where $\theta$is the angle that the door is open, I is the moment of inertia of the door about its hinges, $b > 0$ is a damping constant that varies with the amount of friction on the door, $k > 0$ is the spring constant associated with the swinging door, θ_0 is the initial angle that the door is opened, and u_0 is the initial angular velocity imparted to the door. If I and k are fixed, determine for which values of b the door will not continually swing back and forth when closing.

Solution: Substituting $y(t) = e^{rt}$ as usual we arrive at the characteristic equation for r:

$$Ir^2 + br + k = 0 \Rightarrow r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4Ik}}{2I}$$

It is clear that to avoid oscillations, we must avoid complex roots. The limiting value of b for which oscillations will not happen is the critical value (boundary between overdamped and underdamped cases), i.e. when the discriminant of the quadratic vanishes:

$$b^2 - 4Ik = 0 \Rightarrow b_{critical} = 2\sqrt{Ik}$$

when $r = -b/2I$ and all values of $b \geq b_{critical}$ give non-oscillatory motion.