Solutions, 316-VII

February 13, 2003

1 Problem 4.11.5

The motion of a mass-spring system with damping is governed by

$$y'' + 10y' + ky = 0 \ ; \ y(0) = 1 \ , \ y'(0) = 0 \ .$$

Find the equation of motion and sketch its graph for $k = 20, 25, 30$. **Solution:**

$$\begin{array}{rcl} ky & = & ke^{rt} \\ + & & + \\ 10y' & = & 10re^{rt} \\ + & & + \\ y'' & = & r^2e^{rt} \\ = & & = \\ 0 & = & (r^2 + 10r + k)e^{rt} \end{array}$$

and factoring:

$$r_{\pm} = \frac{-10 \pm \sqrt{100 - 4k}}{2} = -5 \pm \sqrt{25 - k}$$

so that the equation has two real, distinct roots for $k < 25$, two real equal roots (double root) for $k = 25$ and two complex conjugate roots for $k > 25$. Therefore the solutions for the 3 cases given ($k = 20, 25, 30$) are:

$$k = 20 \Rightarrow r_+ = -5 + \sqrt{5} \ , \ r_- = -5 - \sqrt{5}$$

$$
\begin{aligned}
y(t) &= Ae^{-(5-\sqrt{5})t} + Be^{-(5+\sqrt{5})t} \quad ; \quad y(0)=1 \\
y'(t) &= -A(5-\sqrt{5})e^{-(5-\sqrt{5})t} - B(5+\sqrt{5})e^{-(5+\sqrt{5})t} \quad ; \quad y'(0)=0 \\
A+B=1\ ,\ -A(5-\sqrt{5}) - B(5+\sqrt{5}) &= -5(A+B)+\sqrt{5}(A-B)=0 \\
&\Rightarrow A = \frac{1+\sqrt{5}}{2}\ ,\ B = \frac{1-\sqrt{5}}{2} \\
y(t) &= \frac{1+\sqrt{5}}{2}e^{-(5-\sqrt{5})t} + \frac{1-\sqrt{5}}{2}e^{-(5+\sqrt{5})t} \quad .
\end{aligned}
$$

$$
\begin{aligned}
k=25 &\Rightarrow r_\pm = -5(\text{double root}) \\
y(t) &= Ae^{-5t} + Bte^{-5t} \quad ; \quad y(0)=1 \\
y'(t) &= -5Ae^{-5t} + Be^{-5t}(1-5t) \quad ; \quad y'(0)=0 \\
A=1\ ,\ -5A+B=0 &\Rightarrow A=1\ ,\ B=5 \\
y(t) &= e^{-5t} + 5te^{-5t} \quad .
\end{aligned}
$$

$$
\begin{aligned}
k=30 &\Rightarrow r_\pm = -1 \pm i\sqrt{5} \\
y(t) &= Ae^{-t}\cos\left(\sqrt{5}t\right) + Be^{-t}\sin\left(\sqrt{5}t\right)\ ;\ y(0)=1\ ,\ y'(0)=0 \\
y'(t) &= Ae^{-t}\left(-\cos\left(\sqrt{5}t\right) - \sqrt{5}\sin\left(\sqrt{5}t\right)\right) + Be^{-t}\left(-\sin\left(\sqrt{5}t\right) + \sqrt{5}\cos\left(\sqrt{5}t\right)\right) \\
A=1\ ,\ -A+\sqrt{5}B=0 &\Rightarrow A=1\ ,\ B=\frac{\sqrt{5}}{5} \\
y(t) &= e^{-t}\cos\left(\sqrt{5}t\right) + \frac{\sqrt{5}}{5}e^{-t}\sin\left(\sqrt{5}t\right)
\end{aligned}
$$

We now use Maple to plot all three functions for $0 \le t \le 2\pi$.

```
> restart:
```

$$m := 1$$
$$Om0 := 5.$$

```
> Y1 := t -> (1+sqrt(5))*exp(-(5-sqrt(5))*t/2)/2 +
> (1-sqrt(5))*exp(-(5+sqrt(5))*t/2)/2;
```

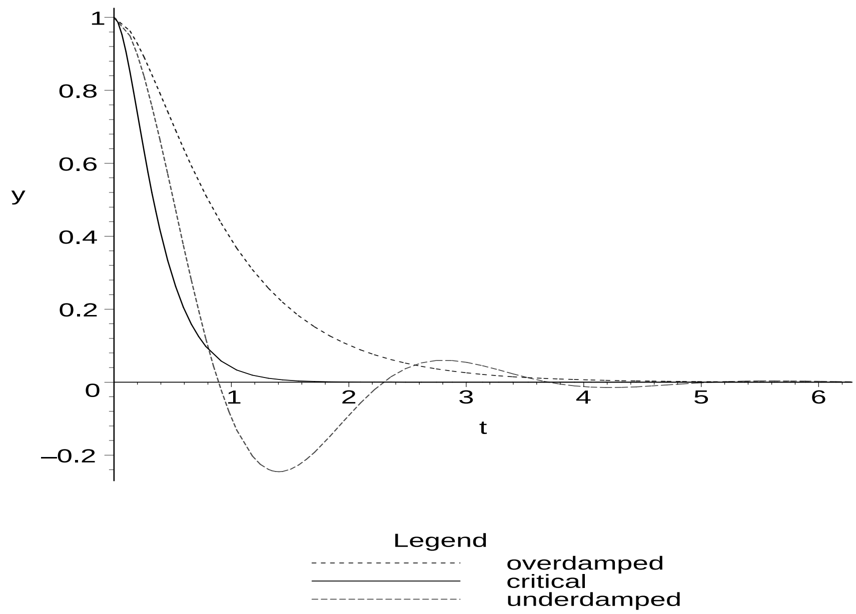
$$Y1 := t \to \frac{1}{2}\,(1+\sqrt{5})\,e^{(-1/2\,(5-\sqrt{5})\,t)} + \frac{1}{2}\,(1-\sqrt{5})\,e^{(-1/2\,(5+\sqrt{5})\,t)}$$

```
> Y2 := t -> exp(-5*t)*(1 + 5*t);  Y3 := t -> exp(-t)*(cos(sqrt(5)*t)
+
> (1/sqrt(5))*sin(sqrt(5)*t));
```

$$Y2 := t \to e^{(-5\,t)}\,(1 + 5\,t)$$

$$Y3 := t \to e^{(-t)}\,(\cos(\sqrt{5}\,t) + \frac{\sin(\sqrt{5}\,t)}{\sqrt{5}})$$

```
> plot([Y1(t),Y2(t),Y3(t)], t =
> 0..2*Pi,color=[blue,black,red],labels=[t,y],legend=[overdamped,critica
> l,underdamped],linestyle=[DASHDOT,SOLID,DASH]);
```



2 Problem 4.11.7

A 1/8-kg mass is attached to a spring with stiffness 16 N/m. The damping constant for the system is 2 N-sec/m. If the mass is pulled 75 cm to the right of equilibrium and given an initial leftward velocity of 2 m/sec, determine the equation of motion of the mass and give its damping factor, quasiperiod and quasifrequency.

Solution: We must solve:

$$\frac{1}{8}y'' + 2y' + 16y = 0 \ ,$$

The roots of the characteristic equation ($r^2 + 16r + 128 = 0$) are

$$r_{\pm} = u \pm iv = -8 \pm i8$$

so that the general solution is

$$y(t) = e^{-8t}\left(C_1 \cos(8t) + C_2 \sin(8t)\right) \ .$$

Solving the IVP we find $C_1 = -.75$, $C_2 = -1$, and the solution

$$y(t) = -\frac{1}{4}e^{-8t}\left(3\cos(8t) + 4\sin(8t)\right) \ .$$

Since $3^2 + 4^2 = 5^2$ we have

$$y(t) = -\frac{5}{4}e^{-8t}\left(\frac{3}{5}\cos(8t) + \frac{4}{5}\sin(8t)\right) = -\frac{5}{4}e^{-8t}\cos(8t - \phi) \ ,$$

where

$$\phi = \tan^{-1}\left(\frac{4}{3}\right) \ .$$

The damping factor is e^{-8t}. The quasiperiod is

$$T = \frac{\pi}{4} \ ,$$

so that the quasifrequency is

$$\nu = \frac{4}{\pi} \ .$$

3 Problem 4.11.11

A 1-kg mass is attached to a spring with stiffness 100 N/m. The damping constant for the system is 0.2 N-sec/m. If the mass is pushed rightward from the equilibrium position with a velocity of 1 m/sec, when will it attain its maximum displacement to the right?

Solution: We must solve:

$$y'' + .2y' + 100y = 0 \ ,$$

The roots of the characteristic equation $(r^2 + .2r + 100 = 0)$ are

$$r_{\pm} = u \pm iv = -.1 \pm i\frac{\sqrt{9999}}{10}$$

so that the general solution is

$$y(t) = e^{ut}\left(C_1 \cos(vt) + C_2 \sin(vt)\right) \ .$$

Solving the IVP we find $C_1 = 0$, $C_2 = .1$, and the solution

$$y(t) = \frac{1}{10} e^{ut} \sin(vt) \ .$$

For the maximum, solve

$$y'(t) = \frac{1}{10} e^{ut}\left(u \sin(vt) + v \cos(vt)\right) = 0$$

to find

$$\tan(vt) = -\frac{v}{u} = 10\frac{\sqrt{9999}}{10} = \sqrt{9999}$$

i.e.

$$t = \frac{10}{\sqrt{9999}} \tan^{-1} \sqrt{9999} \approx 0.15608742057822$$

4 Problem 4.2.13b

Given that $y_1(x) = e^{2x}\cos x$ and $y_2(x) = e^{2x}\sin x$ are solutions to the homogeneous equation

$$y'' - 4y' + 5y = 0 \ ,$$

find solutions to this equation that satisfy the following initial conditions:
(b) $y(\pi) = 4e^{2\pi}$, $y'(\pi) = 5e^{2\pi}$.
Solution:
We have the general solution:

$$y(x) = C_1 e^{2x}\cos x + C_2 e^{2x}\sin x$$

so that

$$y'(x) = C_1\left(2e^{2x}\cos x - e^{2x}\sin x\right) + C_2\left(2e^{2x}\sin x + e^{2x}\cos x\right)$$

and setting $x = \pi$:

$$\begin{aligned} y(\pi) &= C_1 e^{2\pi}\cos\pi + C_2 e^{2\pi}\sin\pi = -C_1 e^{2\pi} = 4e^{2\pi} \\ y'(\pi) &= C_1 e^{2\pi}(2\cos\pi - \sin\pi) + C_2 e^{2\pi}(2\sin\pi + \cos\pi) = -2C_1 e^{2\pi} - C_2 e^{2\pi} = 5e^{2\pi} \end{aligned}$$

so that solving:

$$C_1 = -4 \ , \ C_2 = 3 \ .$$

Then, the solution is:

$$y(x) = e^{2x}(-4\cos x + 3\sin x) \ .$$

This can be written in compact form if we let

$$\cos\phi = \frac{-4}{5} \ , \ \sin\phi = \frac{3}{5}$$

or

$$\phi = \tan^{-1}\left(\frac{3}{-4}\right)$$

then

$$y(x) = 5e^{2x}\cos(x - \phi) \ .$$

5 Problem 4.2.14a

Given that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ are solutions to the homogeneous equation

$$y'' - y' - 2y = 0 \ ,$$

find solutions to this equation that satisfy the following initial conditions:
(a) $y(0) = -1$, $y'(0) = 4$.
Solution:
We have the general solution:

$$y(x) = C_1 e^{2x} + C_2 e^{-x}$$

so that

$$y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$$

and setting $x = 0$:

$$\begin{aligned} y(0) &= C_1 + C_2 = -1 \\ y'(0) &= 2C_1 - C_2 = 4 \end{aligned}$$

so that solving:

$$C_1 = 1 \ , \ C_2 = -2 \ .$$

Then, the solution is:

$$y(x) = e^{2x} - 2e^{-x} \ .$$